# THE GALOIS THEORY OF MATRIX $C$-RINGS

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ABSTRACT. A theory of monoids in the category of bicomodules of a coalgebra $C$ or $C$-rings is developed. This can be viewed as a dual version of the coring theory. The notion of a matrix ring context consisting of two bicomodules and two maps is introduced and the corresponding example of a $C$-ring (termed a *matrix $C$-ring*) is constructed. It is shown that a matrix ring context can be associated to any bicomodule which is a one-sided quasi-finite injector. Based on this, the notion of a *Galois module* is introduced and the structure theorem, generalising Schneider's Theorem II [H.-J. Schneider, Israel J. Math., 72 (1990), 167–195], is proven. This is then applied to the $C$-ring associated to a weak entwining structure and a structure theorem for a weak $A$-Galois coextension is derived. The theory of matrix ring contexts for a firm coalgebra (or *infinite matrix ring contexts*) is outlined. A Galois connection associated to a matrix $C$-ring is constructed.

## 1. INTRODUCTION

The present paper is a contribution to the long standing programme (motivated by non-commutative geometry) of understanding the origins and finding the most general formulation of Schneider's structure theorems for Galois-type extensions [27]. With the re-birth of interest in corings triggered by [6] it has become clear that the proper general formulation of Schneider's Theorem I can be provided by corings and their comodules, and such formulations were achieved in recent papers [7], [17], [31], [4]. It had earlier been realised in [6] that to obtain a generalisation of Schneider's Theorem II, which can be understood as a dual version of Theorem I, one needs to develop new algebraic structures, termed *$C$-rings* in [6, Section 6]. Given a coalgebra $C$ (over a field $k$), a *$C$-ring* is a monoid in the category of $C$-bicomodules (with the monoidal structure provided by the cotensor product $-\underset{C}{\Box}-$). Explicitly a $C$-ring is a $C$-bicomodule $\mathscr{A}$ together with two bicomodule maps $\mu_{\mathscr{A}}:\mathscr{A}\underset{C}{\Box}\mathscr{A}\to\mathscr{A}$ and $\eta_{\mathscr{A}}:C\to\mathscr{A}$ such that

$$\mu_{\mathscr{A}}\circ(\mu_{\mathscr{A}}\underset{C}{\Box}\mathscr{A})=\mu_{\mathscr{A}}\circ(\mathscr{A}\underset{C}{\Box}\mu_{\mathscr{A}}),\qquad \mu_{\mathscr{A}}\circ(\mathscr{A}\underset{C}{\Box}\eta_{\mathscr{A}})=\mu_{\mathscr{A}}\circ(\eta_{\mathscr{A}}\underset{C}{\Box}\mathscr{A})=\mathscr{A},$$

where the standard isomorphisms $\mathscr{A}\underset{C}{\Box}C\simeq\mathscr{A}\simeq C\underset{C}{\Box}\mathscr{A}$ provided by the $C$-coactions are implicitly used. The current most general formulation of Schneider's Theorem I involves not so much corings themselves but a special class of their comodules, termed *principal comodules*. Crucial for this formulation is the notion of a *comatrix coring* introduced in [20, Proposition 2.1], i.e. a coring which can be associated to any bimodule, finitely generated and projective on one side. Prompted by this in the present paper we introduce and study *matrix $C$-rings*, which can be associated to any $(D,C)$-bicomodule that is a quasi-finite injector as a $C$-comodule. As the notion of a *quasi-finite injector* is not as familiar as the notion of a finitely generated projective module, in our definition of a matrix $C$-ring we follow the route suggested by [8, Theorem 2.4], and define matrix $C$-rings through *matrix*

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*ring contexts*. The latter have a very natural meaning as adjoint pairs in a bicategory of bicomodules and are very closely related to Morita-Takeuchi contexts [29].

Recall from [6, Section 6] that a right module of a $C$-ring $\mathscr{A}$ is a right $C$-comodule $M$ together with a right $C$-comodule map $\overline{\rho_M}: M\underset{C}{\square}\mathscr{A} \to M$ such that

$$\overline{\rho_M}\circ(\overline{\rho_M}\underset{C}{\square}\mathscr{A}) = \overline{\rho_M}\circ(M\underset{C}{\square}\mu_{\mathscr{A}}), \qquad \overline{\rho_M}\circ(M\underset{C}{\square}\eta_{\mathscr{A}}) = M,$$

where again the standard isomorphism $M\underset{C}{\square}C \simeq M$ provided by the $C$-coaction on $M$ is implicitly used. We introduce the notion of an $\mathscr{A}$*-coendomorphism coalgebra* of a right $\mathscr{A}$-module, quasi-finite and injective as a $C$-comodule. Starting with a right $\mathscr{A}$-module $M$ which is a quasi-finite injector as a $C$-comodule, we are able to construct a matrix $C$-ring. If this $C$-ring is isomorphic to $\mathscr{A}$, then we say that $M$ is a *Galois module*. If, furthermore, $M$ is an injective module of the $\mathscr{A}$-coendomorphism coalgebra, then we say that $M$ is a *principal Galois module*. We then derive the equivalent conditions for $M$ to be a Galois and principal Galois module in Theorem 3.11. This is the main result of the paper, and is a sought generalisation of Schneider's Theorem II. We then construct a Galois connection associated to a matrix $C$-ring. Finally we apply Theorem 3.11 to a $C$-ring associated to a weak entwining structure and obtain a dual version of results in [13]. In particular we prove that, within an invertible weak entwining structure, a coextension of coalgebras by a (left) self-injective algebra has a Galois property, provided the canonical map is injective.

**Notation.** We work over a field $k$. For a coalgebra $C$, the product is denoted by $\Delta_C$ and the counit by $\varepsilon_C$. For a right (resp. left) $C$-comodule $M$ the coaction is denoted by $\rho^M$ (resp. ${}^M\rho$). We use Sweedler's notation for coproducts $\Delta_C(c) = c_{(1)}\otimes c_{(2)}$, for right coactions $\rho^M(m) = m_{[0]}\otimes m_{[1]}$, and for left coactions ${}^M\rho(m) = m_{[-1]}\otimes m_{[0]}$. The cotensor product is denoted by $-\underset{C}{\square}-$. For a $C$-ring $\mathscr{A}$, $\mu_{\mathscr{A}}$ denotes the product (as a map, on elements it is denoted by a juxtaposition), $\eta_{\mathscr{A}}$ is the unit, $\overline{\rho_M}$ (resp. $\overline{{}_M\rho}$) is the $\mathscr{A}$-action on right (resp. left) $\mathscr{A}$-module $M$. The categories of right (resp. left) $\mathscr{A}$-modules and $C$-comodules are denoted by $\mathbf{M}_{\mathscr{A}}$ and $\mathbf{M}^C$ (resp. ${}_{\mathscr{A}}\mathbf{M}$ and ${}^C\mathbf{M}$), while ${}^C\mathbf{M}_A$ denotes the category of right $A$-modules and left $C$-comodules with right $A$-linear coaction.

## 2. Matrix ring contexts

### 2.1. Quasi-finite matrix contexts.

**Definition 2.1.** A *matrix ring context*, $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$, consists of a pair of coalgebras $C$ and $D$, a $(C,D)$-bicomodule $N$, a $(D,C)$-bicomodule $M$ and a pair of bicomodule maps

$$\sigma: C \to N\underset{D}{\square}M, \qquad \tau: M\underset{C}{\square}N \to D$$

such that the diagrams

$$\begin{tikzcd} N\underset{D}{\square}M\underset{C}{\square}N \arrow[d, "N\underset{D}{\square}\tau"'] & C\underset{C}{\square}N \arrow[l, "\sigma\underset{C}{\square}N"'] \\ N\underset{D}{\square}D & N \arrow[l, "\rho^N"] \arrow[u, "{}^N\rho"'] \end{tikzcd} \qquad \begin{tikzcd} M\underset{C}{\square}N\underset{D}{\square}M \arrow[r, "\tau\underset{D}{\square}M"] & D\underset{D}{\square}M \\ M\underset{C}{\square}C \arrow[u, "M\underset{C}{\square}\sigma"] & M \arrow[l, "\rho^M"] \arrow[u, "{}^M\rho"'] \end{tikzcd}$$

commute. The map $\sigma$ is called a *unit* and $\tau$ is called a *counit* of a matrix context.

Since a counit $\tau$ of a matrix ring context is a $D$-bicomodule map, it is fully determined by its *reduced form* $\widehat{\tau} = \varepsilon_D \circ \tau$. The map $\widehat{\tau}$ is called a *reduced counit* of a matrix context. Note that the $D$-bicolinearity of $\tau$ is equivalent to the following property of $\widehat{\tau}$,

$$(D \otimes \widehat{\tau}) \circ ({}^M\rho \otimes N) = (\widehat{\tau} \otimes D) \circ (M \otimes \rho^N). \tag{2.1}$$

In terms of the reduced counit, the commutative diagrams in Definition 2.1 read

$$(N \otimes \widehat{\tau}) \circ (\sigma \mathop{\Box}_C N) \circ {}^N\rho = N, \qquad (\widehat{\tau} \otimes M) \circ (M \mathop{\Box}_C \sigma) \circ \rho^M = M. \tag{2.2}$$

In other words, equations (2.2) mean that $(N \otimes \widehat{\tau}) \circ (\sigma \mathop{\Box}_C N)$ is the identity on $C \mathop{\Box}_C N$, while $(\widehat{\tau} \otimes M) \circ (M \mathop{\Box}_C \sigma)$ is the identity on $M \mathop{\Box}_C C$.

The notion of a matrix context is closely related to that of *pre-equivalence data* or a *Morita-Takeuchi context* introduced in [29, Definition 2.3]. In particular, in view of [29, Theorem 2.5], if one of the maps in a Morita-Takeuchi context is injective, then there is a corresponding matrix ring context. Furthermore, every equivalence data give rise to a matrix ring context. This relationship explains the use of term *context* in Definition 2.1. The use of term *ring* is justified by the following

**Proposition 2.2.** *Let $(C, D, {}^CN^D, {}^DM^C, \sigma, \tau)$ be a matrix ring context. Then $\mathscr{A} := N \mathop{\Box}_D M$ is a $C$-ring with the product and unit*

$$\mu_{\mathscr{A}} = N \mathop{\Box}_D \widehat{\tau} \mathop{\Box}_D M, \qquad \eta_{\mathscr{A}} = \sigma,$$

*where $\widehat{\tau}$ is the reduced counit. Furthermore, $M$ is a right $\mathscr{A}$-module with the action $\widehat{\tau} \mathop{\Box}_D M$ and $N$ is a left $\mathscr{A}$-module with the action $N \mathop{\Box}_D \widehat{\tau}$. The $C$-ring $\mathscr{A}$ is called a* matrix $C$-ring.

*Proof.* By definition, both $\mu_{\mathscr{A}}$ and $\eta_{\mathscr{A}}$ are $C$-bicomodule maps. Since $\tau$ is a $D$-bicomodule map (cf. equation (2.1)), the product $\mu_{\mathscr{A}}$ is well-defined, i.e. $\mu_{\mathscr{A}}(\mathscr{A} \mathop{\Box}_C \mathscr{A}) \subseteq \mathscr{A}$. The associativity of the product $\mu_{\mathscr{A}}$ follows immediately by the $k$-linearity of $\widehat{\tau}$, while equations (2.2) imply that $\eta_{\mathscr{A}} = \sigma$ is the unit for $\mu_{\mathscr{A}}$. The statements about the actions of $\mathscr{A}$ are proven in a similar way. □

**Example 2.3.** As an immediate example of a matrix ring context, consider a coalgebra map $f : C \to D$. Take $M = N = C$, viewed as a $(D,C)$- or $(C,D)$-bicomodule via the map $f$, and define $\sigma = \Delta_C$ and $\tau = f$. Note that $\widehat{\tau} = \varepsilon_D \circ f = \varepsilon_C$. The corresponding matrix $C$-ring is $\mathscr{A} = C \mathop{\Box}_D C$ with the product $\mu_A = C \mathop{\Box}_D \varepsilon_C \mathop{\Box}_D C$ and unit $\Delta_C$.

We now explore the meaning of a matrix context.

**Proposition 2.4.** *If $(C, D, {}^CN^D, {}^DM^C, \sigma, \tau)$ is a matrix ring context, then the cotensor functor $F = - \mathop{\Box}_C N : \mathbf{M}^C \to \mathbf{Vect}_k$ is a left adoint of the tensor functor $G = - \otimes M : \mathbf{Vect}_k \to \mathbf{M}^C$.*

*Proof.* Define natural transformations

$$\varphi : \mathbf{M}^C \to GF, \qquad \varphi_X := (X \mathop{\Box}_C \sigma) \circ \rho^X,$$

$$\nu : FG \to \mathbf{Vect}_k, \qquad \nu_Y := (Y \otimes \varepsilon_D) \circ (Y \otimes \tau) = Y \otimes \widehat{\tau}.$$

We need to show that these morphisms are the unit and counit, respectively, of the adjunction. Take any right $C$-comodule $X$ and compute

$$\begin{aligned}\nu_{F(X)}\circ F(\varphi_X) &= (X\underset{C}{\Box}N\otimes\widehat{\tau})\circ(((X\underset{C}{\Box}\sigma)\circ\rho^X)\underset{C}{\Box}N)\\ &= (X\underset{C}{\Box}N\otimes\widehat{\tau})\circ(X\underset{C}{\Box}\sigma\underset{C}{\Box}N)\circ(X\underset{C}{\Box}{}^N\rho) = X\underset{C}{\Box}N = F(X),\end{aligned}$$

where the second equality follows by the definition of the cotensor product, and the third equality follows by the first of equations (2.2). On the other hand, for all vector spaces $Y$,

$$G(\nu_Y)\circ\varphi_{G(Y)} = (Y\otimes\widehat{\tau}\otimes M)\circ(Y\otimes M\underset{C}{\Box}\sigma)\circ(Y\otimes\rho^M) = Y\otimes M = G(Y),$$

by the second of equations (2.2). Hence the natural transformations $\varphi$ and $\nu$ satisfy the required properties. □

Since, given a matrix ring context $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$, the functor $-\otimes M:\mathbf{Vect}_k\to\mathbf{M}^C$ has a left adjoint, the right $C$-comodule $M$ is a *quasi-finite* comodule (cf. [29, Proposition 1.3]). The left adjoint of $-\otimes M:\mathbf{Vect}_k\to\mathbf{M}^C$ is known as a *co-hom* functor and is denoted by $h_C(M,-):\mathbf{M}^C\to\mathbf{Vect}_k$. By the uniqueness of adjoints, in the case of a matrix ring context, $h_C(M,-)\simeq -\underset{C}{\Box}N$. Since $h_C(M,-)$ has a right adjoint, it is right exact, and since $-\underset{C}{\Box}N$ is left exact, the above isomorphism of functors implies that the cohom functor $h_C(M,-)$ is exact, i.e. the right $C$-module $M$ is an *injector* (cf. [14, Section 12.8]). Note further that $N\simeq h_C(M,C)$. Thus the notion of a ring context necessarily implies that the right $C$-comodule $M$ is a quasi-finite injector. In the next theorem we associate a matrix coring context to a quasi-finite injector.

**Theorem 2.5.** *Let $M$ be a $(D,C)$-bicomodule and suppose that the right $C$-comodule $M$ is a quasi-finite injector. Define $N:=h_C(M,C)$. Then there exist maps $\sigma$ and $\tau$ such that the sixtuple $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$ is a matrix ring context.*

Recall from [29, Section 1.8] (cf. [14, Sections 12.5–12.6]) that if a $(D,C)$-bicomodule is quasi-finite as a right $C$-comodule, then $h_C(M,C)$ is a $(C,D)$-bicomodule with the left $C$-coaction ${}^{h_C(M,C)}\rho := h_C(M,\Delta_C)$ and the right $D$-coaction $\rho^{h_C(M,C)}$ uniquely determined by the condition

$$(h_C(M,C)\otimes{}^M\rho)\circ\varphi_C = (\rho^{h_C(M,C)}\otimes M)\circ\varphi_C,$$

where $\varphi:\mathbf{M}^C\to h_C(M,-)\otimes M$ is the unit of the adjunction. This explains the $(D,C)$-bicomodule structure of $N$ in the theorem. Recall further from [29, Section 1.17] that for a quasi-finite right $C$-comodule $M$, the vector space $E=h_C(M,M)$ is a coalgebra with the coproduct and counit determined uniquely by relations

$$(E\otimes\varphi_M)\circ\varphi_M = (\Delta_E\otimes M)\circ\varphi_M,\qquad (\varepsilon_E\otimes M)\circ\varphi_M = M. \tag{2.3}$$

$E$ is known as the *coendomorphism coalgebra of $M$*. Furthermore, $M$ is an $(E,C)$-bicomodule. In addition if $M$ is a $(D,C)$-bicomodule, then there exists a unique coalgebra map $\pi:E\to D$ such that ${}^M\rho=(\pi\otimes M)\circ\varphi_M$ (cf. [29, Section 1.18]). Explicitly, $\pi:=(\varepsilon_E\otimes D)\circ\rho^E$, where $\rho^E:E\to E\otimes D$ is the right $D$-coaction on $E$ induced by the left $D$-coaction on $M$. The strategy for the proof of Theorem 2.5 is to prove it first for $D=E$ and then to deduce it for all $D$, using the colagebra map $\pi:E\to D$.

**Lemma 2.6.** *Suppose that a right $C$-comodule $M$ is a quasi-finite injector and define $N:=h_C(M,C)$ and $E:=h_C(M,M)$. Then there exist maps $\sigma_E$ and $\tau_E$ such that the sixtuple $(C,E,{}^CN^E,{}^EM^C,\sigma_E,\tau_E)$ is a matrix ring context.*

*Proof.* First recall that $h_C(M,-)$ can be understood as a functor $\mathbf{M}^C \to \mathbf{M}^E$ which is the left adjoint to the cotensor functor $-\underset{E}{\square}M : \mathbf{M}^E \to \mathbf{M}^C$ (cf. [14, Section 12.7]). Since $M^C$ is a quasi-finite injector, $h_C(M,-) \simeq -\underset{C}{\square}N$ (cf. [14, Section 12.8]). Thus there are the unit and counit of adjunction

$$\varphi : \mathbf{M}^C \to -\underset{C}{\square}N\underset{E}{\square}M, \qquad \nu : -\underset{E}{\square}M\underset{C}{\square}N \to \mathbf{M}^E.$$

Define morphisms of right comodules

$$\sigma_E := \varphi_C : C \to C\underset{C}{\square}N\underset{E}{\square}M \simeq N\underset{E}{\square}M, \qquad \tau_E := \nu_E : M\underset{C}{\square}N \simeq E\underset{E}{\square}M\underset{C}{\square}N \to E.$$

To see that $\sigma_E$ is a $C$-bicomodule map use the fact that $\varphi$ is a natural tranformation to produce commutative diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\varphi_C"'] & C\otimes C \arrow[d, "\varphi_{C\otimes C}"] \\ C\underset{C}{\square}N\underset{E}{\square}M \arrow[r, "\Delta_C\underset{C}{\square}N\underset{E}{\square}M"'] & C\otimes C\underset{C}{\square}N\underset{E}{\square}M \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "l_c"] \arrow[d, "\varphi_C"'] & C\otimes C \arrow[d, "\varphi_{C\otimes C}"] \\ C\underset{C}{\square}N\underset{E}{\square}M \arrow[r, "l_c\underset{C}{\square}N\underset{E}{\square}M"'] & C\otimes C\underset{C}{\square}N\underset{E}{\square}M \end{tikzcd}$$

where $l_c(c') = c\otimes c'$, for all $c, c' \in C$. Since $\Delta_C\underset{C}{\square}N\underset{E}{\square}M$ can be identified with the left $C$-coaction ${}^{N\underset{E}{\square}M}\rho$, putting these two diagrams together we obtain, for all $c \in C$,

$$\begin{aligned} {}^{N\underset{E}{\square}M}\rho\circ\varphi_C(c) &= \varphi_{C\otimes C}(c_{(1)}\otimes c_{(2)}) = \varphi_{C\otimes C}\circ l_{c_{(1)}}(c_{(2)}) \\ &= (l_{c_{(1)}}\underset{C}{\square}N\underset{D}{\square}M)\circ\varphi_C(c_{(2)}) = c_{(1)}\otimes\varphi_C(c_{(2)}). \end{aligned}$$

Hence $\sigma_E = \varphi_C$ is a $C$-bicomodule map. A similar method can be used to show that $\tau_E$ is an $E$-bicomodule map. By the properties of the unit and counit of adjunction, the composition

$$C\underset{C}{\square}N \xrightarrow{\varphi_C\underset{C}{\square}N} C\underset{C}{\square}N\underset{E}{\square}M\underset{C}{\square}N \xrightarrow{\nu_{C\underset{C}{\square}N}} C\underset{C}{\square}N \tag{2.4}$$

yields the identity. Since $\nu$ is a natural transformation, the commutative diagrams induced by the morphisms $\rho^N$, $l_n : E \to N\otimes E$, $x \mapsto n\otimes x$, and ${}^N\rho$, give the following equalities

$$\nu_{N\otimes E}\circ(\rho^N\underset{E}{\square}M\underset{C}{\square}N) = \rho^N\circ\nu_N \tag{2.5}$$

$$\nu_{N\otimes E}\circ(l_n\underset{E}{\square}M\underset{C}{\square}N) = l_n\circ\nu_E \tag{2.6}$$

respectively. Hence, for all $n\otimes m\otimes n' \in N\underset{E}{\square}M\underset{C}{\square}N$ (summation suppressed for simplicity),

$$\begin{aligned} \rho^N\circ\nu_N(n\otimes m\otimes n') &= \nu_{N\otimes E}\circ(n_{[0]}\otimes n_{[1]}\otimes m\otimes n') = \nu_{N\otimes E}\circ(l_{n_{[0]}}(n_{[1]})\otimes m\otimes n') \\ &= l_{n_{[0]}}\circ\nu_E(n_{[1]}\otimes m\otimes n') = n_{[0]}\otimes\nu_E(n_{[1]}\otimes m\otimes n'), \end{aligned}$$

where the first equality is from (2.5) and last by (2.6). And so applying $N\otimes\varepsilon_E$ to both sides and using the canonical identification $N\underset{E}{\square}E \simeq N$, we obtain $\nu_N = N\otimes\widehat{\tau}_E$, where $\widehat{\tau}_E := \varepsilon_E\circ\tau_E$. Since $\sigma_E = \varphi_C$, the first of relations (2.2) follows by the fact that the composition (2.4) is the identity. The other condition in (2.2) is proven in a similar way. $\square$

Note that the map $\tau_E$ constructed in the proof of Lemma 2.6 is a bijection, hence $(C, E, {}^CN^E, {}^EM^C, \sigma_E, \tau_E^{-1})$ is a Morita-Takeuchi context.

In the situation of Theorem 2.5, the coalgebra map $\pi : E \to D$ induces the map $N\underset{E}{\square}M \to N\underset{D}{\square}M$. Using the matrix ring context $(C,E,{}^CN^E,{}^EM^C,\sigma_E,\tau_E)$ in Lemma 2.6, define the required matrix ring context $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$ by

$$\sigma : C \xrightarrow{\sigma_E} N\underset{E}{\square}M \to N\underset{D}{\square}M, \qquad \tau : M\underset{C}{\square}N \xrightarrow{\tau_E} E \xrightarrow{\pi} D.$$

This completes the proof of Theorem 2.5.

The notion of a matrix ring context has a very natural interpretation in the language of bicategories. Consider the bicategory of bicomodules where 0-cells are coalgebras, 1-cells are bicomodules and 2-cells are bicomodule maps. Define the composite, $g\circ f$, of two 1-cells $f : X \to Y$ and $g : Y \to Z$ to be $f\underset{Y}{\square}g : X \to Z$. Then there are obvious associativity and unit isomorphisms. When the isomorphisms implicitly used in Definition 2.1, such as $(N\underset{D}{\square}M)\underset{C}{\square}N \cong N\underset{D}{\square}(M\underset{C}{\square}N)$, are fully described it becomes apparent that in this language $(C,D,g : C \to D, f : D \to C,\sigma,\tau)$ is a matrix ring context if and only if the 2-cells $\sigma : 1_C \Rightarrow f\circ g$ and $\tau : g\circ f \Rightarrow 1_D$ form an adjoint pair in the bicategory.

2.2. **Infinite (firm) matrix contexts.** The extension of comatrix coring contexts to non-unital firm rings in [22] (cf. [18], both extending infinite comatrix corings of [21]) allows for a generalisation of matrix ring contexts as in Definition 2.1 whereby one is no longer confined to quasi-finite injectors. We outline basic properties of such a generalisation in the present section.

Let $D$ be a non-counital coalgebra with coproduct $\Delta_D$. We say that $D$ is a *firm coalgebra* if the map $\Delta_D : D \to D\underset{D}{\square}D$ is an isomorphism. The inverse of $\Delta_D$ is denoted by $\nabla_D : D\underset{D}{\square}D \to D$. A left (resp. right) non-unital comodule $M$ of a firm coalgebra $D$ is said to be *firm*, provided the coaction ${}^M\rho : M \to D\underset{D}{\square}M$ (resp. $\rho^M : M \to M\underset{D}{\square}D$) is an isomorphism of comodules. The inverse of coaction is denoted by ${}_M\nabla$ (resp. $\nabla_M$).

**Definition 2.7.** An *infinite matrix ring context*, $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$, consists of a counital coalgebra $C$, firm coalgebra $D$, a $(C,D)$-bicomodule $N$, a $(D,C)$-bicomodule $M$, both counital as $C$-comodules and firm as $D$-comodules, and a pair of bicomodule maps

$$\sigma : C \to N\underset{D}{\square}M, \qquad \tau : M\underset{C}{\square}N \to D$$

such that the diagrams in Definition 2.1 commute.

In contrast to (finite) matrix ring context in Definition 2.1, the counit $\tau$ of an infinite matrix context does not have a reduced form. Following the same line of argument as in [15, Theorem 1.1.3], one can associate a pair of adjoint functors with any infinite matrix ring context.

**Proposition 2.8.** *Given an infinite matrix ring context $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$, denote by $\mathbf{M}^D$ the category of firm right $D$-comodules. Then the functor $F = -\underset{C}{\square}N : \mathbf{M}^C \to \mathbf{M}^D$ is the left adjoint of $G = -\underset{D}{\square}M : \mathbf{M}^D \to \mathbf{M}^C$.*

*Proof.* This can be proven in the same way as Proposition 2.4, provided one replaces all references to $\varepsilon_D$ by the inverses of the coactions such as $\nabla_Y$ etc. The unit of the adjunction is $\varphi : \mathbf{M}^C \to GF$, $\varphi_X = (M\underset{C}{\square}\sigma)\circ\rho^X$, and the counit is $\nu : FG \to \mathbf{M}^C$, $\nu_Y = \nabla_Y \circ (Y\underset{D}{\square}\tau)$. $\square$

Note that this adjoint pair of functors no longer extends to functors $\mathbf{M}^C \to \mathbf{Vect}_k$, $\mathbf{Vect}_k \to \mathbf{M}^C$. Consequently, $M$ is no longer a quasi-finite injector as a right $C$-comodule. Still, associated to an infinite matrix ring context are a $C$-ring and a firm coalgebra. Their construction is very reminiscent of the construction of an *elementary algebra* in the Morita theory of non-unital rings (cf. [15, p. 36], [30, p. 129]).

**Proposition 2.9.** *Let $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$ be an infinite matrix ring context.*

*(1) $\mathscr{A} := N\underset{D}{\Box}M$ is a $C$-ring with the product and unit*

$$\mu_{\mathscr{A}} = (\nabla_N\underset{D}{\Box}M)\circ(N\underset{D}{\Box}\tau\underset{D}{\Box}M) = (N\underset{D}{\Box}{}_M\nabla)\circ(N\underset{D}{\Box}\tau\underset{D}{\Box}M), \qquad \eta_{\mathscr{A}} = \sigma.$$

*(2) $E := M\underset{C}{\Box}N$ is a firm coalgebra with the coproduct*

$$\Delta_E = (M\underset{C}{\Box}\sigma\underset{C}{\Box}N)\circ(\rho^M\underset{C}{\Box}N) = (M\underset{C}{\Box}\sigma\underset{C}{\Box}N)\circ(M\underset{C}{\Box}{}^N\rho).$$

*Proof.* (1) $\nabla_N$ is necessarily a $(C,D)$-bicomodule map, since it is the inverse of a $(C,D)$-bicomodule map $\rho^N$. This means that the map $\mu_{\mathscr{A}}$ is $C$-bicolinear. To see that the two forms of $\mu_{\mathscr{A}}$ are equivalent, apply $\rho^N\underset{D}{\Box}M$ to get $N\underset{D}{\Box}\tau\underset{D}{\Box}M$ in both cases (note that $\rho^N\underset{D}{\Box}M = N\underset{D}{\Box}{}^M\rho$). That $\eta_{\mathscr{A}}$ is the unit for $\mu_{\mathscr{A}}$ follows by commutative diagrams in Definition 2.1, while the associativity of $\mu_{\mathscr{A}}$ is clear from the definition.

(2) The map $M\underset{C}{\Box}\sigma\underset{C}{\Box}N$ is coassociative by the coassociativity of coactions and colinearity of $\sigma$. Define

$$\nabla_E : E\underset{E}{\Box}E \to E, \qquad \nabla_E = ({}_M\nabla\underset{C}{\Box}N)\circ(\tau\underset{D}{\Box}M\underset{C}{\Box}N).$$

Note that $\Delta_E$ is $D$-bicolinear, hence, in particular $E\underset{E}{\Box}E \subseteq E\underset{D}{\Box}E$. Using this, one checks that $\nabla_E$ is the inverse of $\Delta_E$ by a routine calculation. □

Note that the coalgebra $E$ plays the same role as the coendomorphism coalgebra $h_C(M,M)$ in the quasi-finite projector case.

## 3. $\mathscr{A}$-COENDOMORPHISM COALGEBRA AND GALOIS MODULES

The aim of this section is to introduce the notion of a Galois module, to derive the structure theorem for such modules and construct the associated Galois connection. Galois modules are a particular class of modules of a $C$-ring $\mathscr{A}$ that are quasi-finite injectors as $C$-comodules. First we need to introduce the notion of an $\mathscr{A}$-coendomorphism coalgebra.

### 3.1. **The $\mathscr{A}$-coendomorphism coalgebra and $C$-ring.**

**Lemma 3.1.** *Let $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$ be a matrix ring context and let $\mathscr{A}$ be a $C$-ring. If $M$ is a right $\mathscr{A}$-module, via the map $\overline{\rho_M} : M\underset{C}{\Box}\mathscr{A} \to M$, then $N$ is a left $\mathscr{A}$-module via the map*

$$\overline{{}_N\rho} : \mathscr{A}\underset{C}{\Box}N \to N, \qquad \overline{{}_N\rho} := (N\otimes\widehat{\tau})\circ(N\underset{D}{\Box}\overline{\rho_M}\underset{C}{\Box}N)\circ(\sigma\underset{C}{\Box}\mathscr{A}\underset{C}{\Box}N)\circ({}^{\mathscr{A}}\rho\underset{C}{\Box}N)$$

*Proof.* The map $\overline{{}_N\rho}$ is left $C$-colinear because it is a composition of left $C$-colinear maps. We need to show that $\overline{{}_N\rho}$ is associative and unital. Throughout the proof we write

$\sigma(c) = c^{[1]}\otimes c^{[2]} \in N\underset{D}{\Box}M$ (summation assumed). The right action $\overline{\rho_M}$ of $\mathscr{A}$ on $M$ is denoted by $\triangleleft$ between the elements. Similarly, the map $\overline{{}_N\rho}$ is denoted by $\triangleright$. In this notation

$$\sum_i a^i \triangleright n^i = \sum_i {a^i}_{[-1]}{}^{[1]}\widehat{\tau}({a^i}_{[-1]}{}^{[2]} \triangleleft {a^i}_{[0]}\otimes n^i), \qquad \text{for all } \sum_i a^i\otimes n^i \in \mathscr{A}\underset{C}{\Box}N.$$

Take any $a\otimes a'\otimes n \in \mathscr{A}\underset{C}{\Box}\mathscr{A}\underset{C}{\Box}N$ (summation suppressed for clarity), and compute

$$\begin{aligned}
(a\triangleright(a'\triangleright n)) &= a_{[-1]}{}^{[1]}\widehat{\tau}(a_{[-1]}{}^{[2]}\triangleleft a_{[0]}\otimes a'_{[-1]}{}^{[1]})\widehat{\tau}(a'_{[-1]}{}^{[2]}\triangleleft a'_{[0]}\otimes n)\\
&= a_{[-1]}{}^{[1]}\widehat{\tau}(a_{[-1]}{}^{[2]}\triangleleft a_{[0]}\otimes a_{[1]}{}^{[1]})\widehat{\tau}(a_{[1]}{}^{[2]}\triangleleft a'\otimes n)\\
&= a_{[-1]}{}^{[1]}\widehat{\tau}((a_{[-1]}{}^{[2]}\triangleleft a_{[0]})_{[0]}\otimes (a_{[-1]}{}^{[2]}\triangleleft a_{[0]})_{[1]}{}^{[1]})\\
&\quad \times\widehat{\tau}((a_{[-1]}{}^{[2]}\triangleleft a_{[0]})_{[1]}{}^{[2]}\triangleleft a'\otimes n)\\
&= a_{[-1]}{}^{[1]}\widehat{\tau}((a_{[-1]}{}^{[2]}\triangleleft a_{[0]})\triangleleft a'\otimes n) = a_{[-1]}{}^{[1]}\widehat{\tau}((a_{[-1]}{}^{[2]}\triangleleft (a_{[0]}a'))\otimes n)\\
&= (aa')_{[-1]}{}^{[1]}\widehat{\tau}(((aa')_{[-1]}{}^{[2]}\triangleleft (aa')_{[0]})\otimes n) = ((aa')\triangleright n),
\end{aligned}$$

where the second equality holds because $a\otimes a' \in \mathscr{A}\underset{C}{\Box}\mathscr{A}$, the third by the right $C$-colinearity of the right $\mathscr{A}$-action on $M$. The fourth equality comes from the second of equations (2.2). The fifth equality follows because the right $\mathscr{A}$-action is multiplicative and the penultimate equality uses the colinearity of the product $\mu_{\mathscr{A}} : \mathscr{A}\underset{C}{\Box}\mathscr{A} \to \mathscr{A}$. This proves that the map $\overline{{}_N\rho}$ is associative. The unitality of $\overline{{}_N\rho}$ follows by a similar calculation that uses the $C$-colinearity of the unit $\eta_{\mathscr{A}}$ and of $\sigma$, the unitality of $\overline{\rho_M}$ and the first of equations (2.2). $\square$

In the set-up of Lemma 3.1, the left action of matrix $C$-ring $N\underset{D}{\Box}M$ on $N$ induced from the right action described in Proposition 2.2 is $N\underset{D}{\Box}\widehat{\tau}$. The next lemma shows that the $\mathscr{A}$-actions are compatible with the unit and counit of a matrix ring context.

**Lemma 3.2.** *Let $(C,D,{}^CN^D,{}^DM^C,\sigma,\tau)$ be a matrix ring context and let $\mathscr{A}$ be a $C$-ring. Suppose that $M$ is a right $\mathscr{A}$-module, via the map $\overline{\rho_M} : M\underset{C}{\Box}\mathscr{A} \to M$ (denoted by $\triangleleft$ between elements) and let $\overline{{}_N\rho}$ be the left $\mathscr{A}$-action on $N$ constructed in Lemma 3.1 (denoted by $\triangleright$ between elements).*

*(1) For all $m\otimes a\otimes n \in M\underset{C}{\Box}\mathscr{A}\underset{C}{\Box}N$ (summation suppressed for clarity),*

$$\widehat{\tau}(m\triangleleft a\otimes n) = \widehat{\tau}(m\otimes a\triangleright n).$$

*(2) The following diagram*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\rho^{\mathscr{A}}"] \arrow[d, "{}^{\mathscr{A}}\rho"'] & \mathscr{A}\underset{C}{\Box}C \arrow[r, "\mathscr{A}\underset{C}{\Box}\sigma"] & \mathscr{A}\underset{C}{\Box}N\underset{D}{\Box}M \arrow[d, "\overline{{}_N\rho}\underset{D}{\Box}M"] \\
C\underset{C}{\Box}\mathscr{A} \arrow[r, "\sigma\underset{C}{\Box}\mathscr{A}"] & N\underset{D}{\Box}M\underset{C}{\Box}\mathscr{A} \arrow[r, "N\underset{D}{\Box}\overline{\rho_M}"] & N\otimes M
\end{tikzcd}$$

*is commutative.*

*Proof.* Both statements follow by straightforward calculations which use the definition of $\overline{{}_N\rho}$, the second of equations (2.2) and the definition of a cotensor product (in the case of assertion (1)), and the $C$-colinearity of $\overline{\rho_M}$ (in the case of assertion (2)). $\square$

**Theorem 3.3.** *Let $\mathscr{A}$ be a $C$-ring and $M$ a right $\mathscr{A}$-module which is a quasi-finite injector as a right $C$-comodule. Let $N := h_C(M,C)$, $E := h_C(M,M)$ and define a vector space $E_{\mathscr{A}}(M)$ as the coequaliser*

$$M \underset{C}{\Box} \mathscr{A} \underset{C}{\Box} N \underset{M \underset{C}{\Box} \overline{{}_N\rho}}{\overset{\overline{\rho_M} \underset{C}{\Box} N}{\rightrightarrows}} M \underset{C}{\Box} N \xrightarrow{\quad \pi_{\mathscr{A}} \quad} E_{\mathscr{A}}(M),$$

*where $\overline{\rho_M}$ is the right $\mathscr{A}$-action on $M$ and $\overline{{}_N\rho}$ is the induced left $\mathscr{A}$-action on $N$ as in Lemma 3.1 corresponding to the matrix ring context $(C, E, {}^CN^E, {}^EM^C, \sigma_E, \tau_E)$ in Lemma 2.6. Then $E_{\mathscr{A}}(M)$ is a coalgebra such that*

$$E \simeq M \underset{C}{\Box} N \xrightarrow{\quad \pi_{\mathscr{A}} \quad} E_{\mathscr{A}}(M)$$

*is a coalgebra map. The coalgebra $E_{\mathscr{A}}(M)$ is called an $\mathscr{A}$-*coendomorphism coalgebra of *$M$.*

*Proof.* Since $M^C$ is a quasi-finite injector, $E$ is isomorphic to $M \underset{C}{\Box} N$. The induced coproduct and counit in $M \underset{C}{\Box} N$ are $M \underset{C}{\Box} \sigma_E \underset{C}{\Box} N$ and $\widehat{\tau}_E$. Lemma 3.2 implies that these two maps factor through the coequaliser defining $E_{\mathscr{A}}(M)$ and hence provide the latter with the coalgebra structure such that $\pi_{\mathscr{A}}$ is a coalgebra map. □

**Corollary 3.4.** *Let $\mathscr{A}$ be a $C$-ring and $M$ a right $\mathscr{A}$-module which is a quasi-finite injector as a $C$-comodule and let $N := h_C(M,C)$. Denote the induced left $C$-coaction on $N$ by ${}^N\rho$. Then*

*(1) $M$ is an $(E_{\mathscr{A}}(M), C)$ - bicomodule, with left coaction $(\pi_{\mathscr{A}} \otimes M) \circ (M \underset{C}{\Box} \sigma_E) \circ \rho^M$. Furthermore this left coaction is right $\mathscr{A}$-linear.*

*(2) $N$ is a $(C, E_{\mathscr{A}}(M))$ - bicomodule, with right coaction $(N \otimes \pi_{\mathscr{A}}) \circ (\sigma_E \underset{C}{\Box} N) \circ {}^N\rho$. Furthermore this right coaction is left $\mathscr{A}$-linear.*

*Proof.* That $M$ is a bicomodule with these coactions follows immediately from the facts that $M$ is a left comodule of $h_C(M,M)$ (with the coaction $(M \underset{C}{\Box} \sigma_E) \circ \rho^M$) and that $\pi_{\mathscr{A}}$ in Theorem 3.3 is a coalgebra map. That the left coaction is right $\mathscr{A}$-linear follows from the defining property of $\pi_{\mathscr{A}}$ and Lemma 3.2. The second part of the corollary is proved in a similar way. □

Thus to any right $\mathscr{A}$-module $M$ which is a quasi-finite injector as a right $C$-comodule one can associate the matrix ring context $(C, E_{\mathscr{A}}(M), {}^CN^{E_{\mathscr{A}}(M)}, {}^{E_{\mathscr{A}}(M)}M^C, \sigma, \tau)$ as in the proof of Theorem 2.5, i.e. with

$$\sigma : C \xrightarrow{\sigma_E} N \underset{E}{\Box} M \to N \underset{E_{\mathscr{A}}(M)}{\Box} M, \qquad \tau : M \underset{C}{\Box} N \xrightarrow{\tau_E} E \xrightarrow{\pi_{\mathscr{A}}} E_{\mathscr{A}}(M).$$

We refer to this context as an *$\mathscr{A}$-coendomorphism context* associated to $M$. The corresponding matrix $C$-ring is referred to as an *$\mathscr{A}$-coendomorphism ring of $M$*.

3.2. **Galois and principal modules.** The aim of this subsection is to study the relationship between $\mathscr{A}$ and the $\mathscr{A}$-coendomorphism ring of $M$.

**Proposition 3.5.** *Let $\mathscr{A}$ be a $C$-ring and $M$ a right $\mathscr{A}$-module which is a quasi-finite injector as a $C$-comodule. Set $N := h_C(M,C)$ and define a map*

$$\beta : \mathscr{A} \to N \otimes M, \qquad \beta := (N \otimes \overline{\rho_M}) \circ (\sigma \underset{C}{\Box} \mathscr{A}) \circ {}^{\mathscr{A}}\rho,$$

*where $\overline{\rho_M}$ denotes the $\mathscr{A}$-action on $M$ and $\sigma$ is the unit of the $\mathscr{A}$-coendomorphism context associated to $M$. Write $S$ for the coalgebra $E_{\mathscr{A}}(M)$. Then:*

*(1) $\beta(\mathscr{A}) \subset N \underset{S}{\Box} M$.*

*(2) The map $\beta$ is a morphism of $C$-rings.*

*Proof.* (1) Write $\sigma(c) = c^{[1]} \otimes c^{[2]}$, for all $c \in C$. Note that on elements $\sigma(c) = \sigma_E(c)$, hence we use the same notation for $\sigma_E$. Writing $\triangleleft$ for the right action of $\mathscr{A}$ on $M$, the map $\beta$ takes the following explicit form, $\beta(a) = {a_{[-1]}}^{[1]} \otimes {a_{[-1]}}^{[2]} \triangleleft a_{[0]}$, for all $a \in \mathscr{A}$. Denote the left (resp. right) $S$-coaction on $M$ (resp. $N$) in Corollary 3.4 by ${}^M\rho$ (resp. $\rho^N$). Then

$$\begin{aligned}
(N \otimes {}^M\rho)(\beta(a)) &= {a_{[-1]}}^{[1]} \otimes \pi_{\mathscr{A}}({a_{[-1]}}^{[2]} \triangleleft a_{[0]} \otimes {a_{[1]}}^{[1]}) \otimes {a_{[1]}}^{[2]} \\
&= {a_{[-1]}}^{[1]} \otimes \pi_{\mathscr{A}}({a_{[-1]}}^{[2]} \otimes a_{[0]} \triangleright {a_{[1]}}^{[1]}) \otimes {a_{[1]}}^{[2]} \\
&= {a_{[-2]}}^{[1]} \otimes \pi_{\mathscr{A}}({a_{[-2]}}^{[2]} \otimes {a_{[-1]}}^{[1]}) \otimes {a_{[-1]}}^{[2]} \triangleleft a_{[0]} = (\rho^N \otimes M)(\beta(a)),
\end{aligned}$$

where the first equality follows by the right $C$-colinearity of the $\mathscr{A}$-action, the second one is the defining property of $\pi_{\mathscr{A}}$. The third equality follows by Lemma 3.2(2) and to derive the last equality, the left $C$-colinearity of $\sigma$ was used.

(2) The map $\beta$ is left $C$-colinear by the left colinearity of $\sigma$. It is right $C$-colinear by the right $C$-colinearity of the $\mathscr{A}$-action $\overline{\rho_M}$. To check that $\beta$ is a unital map, take any $c \in C$ and compute

$$\begin{aligned}
\beta \circ \eta_{\mathscr{A}}(c) &= (N \underset{D}{\Box} \overline{\rho_M}) \circ (\sigma \underset{C}{\Box} \mathscr{A}) \circ {}^{\mathscr{A}}\rho \circ \eta_{\mathscr{A}}(c) = (N \underset{D}{\Box} \overline{\rho_M}) \circ (\sigma \underset{C}{\Box} \mathscr{A})(c_{(1)} \otimes \eta_{\mathscr{A}}(c_{(2)})) \\
&= (N \underset{D}{\Box} \overline{\rho_M})(c^{[1]} \otimes {c^{[2]}}_{[0]} \otimes \eta_{\mathscr{A}}({c^{[2]}}_{[1]})) = c^{[1]} \otimes \overline{\rho_M} \circ (M \underset{C}{\Box} \eta_{\mathscr{A}}) \circ \rho^M(c^{[2]}) = \sigma(c),
\end{aligned}$$

where the second equality is by the left $C$-colinearity of $\eta_{\mathscr{A}}$, the third equality is by the left $C$-colinearity of $\sigma$ and the final equality is by the unitality of a right $\mathscr{A}$-action. Since $\sigma$ is the unit map for the $\mathscr{A}$-coendomorphism $C$-ring $N \underset{S}{\Box} M$, $\beta$ is a unital map as required. A calculation, virtually the same as that proving the associativity of the left $\mathscr{A}$-action in the proof of Lemma 3.1, confirms that $\beta$ is a multiplicative map too. □

**Definition 3.6.** Take $M \in \mathbf{M}_{\mathscr{A}}$ such that $M^C$ is a quasi-finite injector, set $N = h_C(M,C)$ and let $S := E_{\mathscr{A}}(M)$ be the $\mathscr{A}$-coendomorphism coalgebra of $M$. We say that $M$ is a *Galois $\mathscr{A}$-module* iff the map $\beta : \mathscr{A} \to N \underset{S}{\Box} M$ in Proposition 3.5 is bijective. A Galois $\mathscr{A}$-module $M$ is said to be *principal* iff $M$ is injective as a left $S$-comodule.

The notion of a Galois module generalises that of a Galois $C$-ring introduced in [6, Section 6]. To make this statement more transparent we recall a lemma and definition from [6, Section 6].

**Lemma 3.7.** *For any $C$-ring $\mathscr{A}$, there is a bijective correspondence between right $\mathscr{A}$-actions, $\overline{\rho_C} : C \underset{C}{\Box} \mathscr{A} \to C$, and nontrivial characters $\kappa : \mathscr{A} \to k$. Here by a nontrivial character we mean a map $\kappa : \mathscr{A} \to k$ which is multiplicative and satisfies $\kappa \circ \eta_{\mathscr{A}} = \varepsilon_C$.*

*Proof.* The correspondence is given as follows: given a right $\mathscr{A}$-action $\overline{\rho_C}$, the corresponding character is given by $\kappa[\overline{\rho_C}] := \varepsilon_C \circ \overline{\rho_C} \circ {}^{\mathscr{A}}\rho$. In the other direction, for each character $\kappa$ there is a map $\overline{\rho_C}[\kappa]$ defined as $\overline{\rho_C}[\kappa](c \otimes a) = \varepsilon_C(c)\kappa(a_{[0]})a_{[1]}$. $\square$

In the case that $\mathscr{A}$ has a nontrivial character, we can study the set

$$I_\kappa = \{\kappa(a_{[0]})a_{[1]} - a_{[-1]}\kappa(a_{[0]}) | a \in \mathscr{A}\} \subseteq C,$$

which is easily checked to be a coideal. Hence we are able to define a coalgebra of coinvariants $B_\kappa = C/I_\kappa$.

**Definition 3.8.** A $C$-ring $\mathscr{A}$ with a nontrivial character $\kappa$ is called a *Galois $C$-ring* if there exists an isomorphism of $C$-rings $\beta : \mathscr{A} \to C \underset{B_\kappa}{\square} C$ such that $\kappa = (\varepsilon_C \underset{B_\kappa}{\square} \varepsilon_C) \circ \beta$.

**Proposition 3.9.** *If $C$ is a Galois module for some $C$-ring $\mathscr{A}$, then $\mathscr{A}$ is a Galois $C$-ring.*

*Proof.* Note that $C^C$ is a quasi-finite injector: $(C, C, {}^C C^C, {}^C C^C, \sigma, \tau)$ is a matrix ring context, where $\tau : C \underset{C}{\square} C \to C$ is the obvious isomorphism and $\sigma = \Delta_C$, corresponding to the identity map $C \to C$ as in Example 2.3. Obviously, $C = h_C(C, C)$. Since $C$ has a right $\mathscr{A}$-action it also has a non-trivial character $\kappa$, provided by the 1-1 correspondence in Lemma 3.7. In terms of this character the right $\mathscr{A}$-action is, for all $c \otimes a \in C \underset{C}{\square} \mathscr{A}$, $\overline{\rho_C}(c \otimes a) = \varepsilon_C(c)\kappa(a_{[0]})a_{[1]}$, so, for all $a \in \mathscr{A}$,

$$\beta(a) = (N \underset{D}{\square} \overline{\rho_C}) \circ (\sigma \underset{C}{\square} \mathscr{A}) \circ {}^{\mathscr{A}}\rho(a) = a_{[-2]} \otimes a_{[-1]} \lhd a_{[0]} = a_{[-1]} \otimes \kappa(a_{[0]})a_{[1]}.$$

Hence $\kappa = (\varepsilon_C \underset{D}{\square} \varepsilon_C) \circ \beta$. Feeding the above explicit form of the right $\mathscr{A}$-action on $C$ into Lemma 3.1, we obtain a left $\mathscr{A}$-action on $C$, $\overline{{}_C\rho}(a \otimes c) = a_{[-1]}\kappa(a_{[0]})\varepsilon_C(c)$, for all $a \otimes c \in \mathscr{A} \underset{C}{\square} C$. Thus

$$S = C \underset{C}{\square} C / \mathrm{Im}(\overline{\rho_C} \underset{C}{\square} C - C \underset{C}{\square} \overline{{}_C\rho}) \simeq C / \{\kappa(a_{(0)})a_{(1)} - a_{(-1)}\kappa(a_{(0)}) | a \in \mathscr{A}\} = B_\kappa.$$

Hence $\beta : \mathscr{A} \to C \underset{B_\kappa}{\square} C$ makes $\mathscr{A}$ into a Galois $C$-ring. $\square$

Following a similar line of argument as in [31, Section 4.8] one proves

**Proposition 3.10.** *If $M$ is a right principal Galois module of a $C$-ring $\mathscr{A}$, then $\mathscr{A}$ is an injective left $C$-comodule.*

*Proof.* Suppose that $M$ is a principal Galois $\mathscr{A}$-module, write $N = h_C(M, C)$ and $S = E_{\mathscr{A}}(M)$, and let $E = M \underset{C}{\square} N$ denote the $C$-coendomorphism coalgebra of $M$. Since $\mathscr{A} \simeq N \underset{S}{\square} M$, there is a chain of isomorphisms

$$N \underset{E}{\square} M \underset{C}{\square} \mathscr{A} \simeq N \underset{E}{\square} M \underset{C}{\square} N \underset{S}{\square} M \simeq N \underset{E}{\square} E \underset{S}{\square} M \simeq N \underset{S}{\square} M \simeq \mathscr{A}.$$

Explicitly the isomorphism $\mathscr{A} \to N \underset{E}{\square} M \underset{C}{\square} \mathscr{A}$ is $(\sigma_E \otimes \mathscr{A}) \circ {}^{\mathscr{A}}\rho$, where $\sigma_E$ is the unit of the matrix ring context in Lemma 2.6. Since $M$ is an injective left $S$-comodule and $M \underset{C}{\square} \mathscr{A} \simeq M \underset{C}{\square} N \underset{S}{\square} M = E \underset{S}{\square} M$, $M \underset{C}{\square} \mathscr{A}$ is injective as a left $E$-comodule. Thus there exists a left $E$-comodule retraction $p$ of the obvious inclusion $\iota : M \underset{C}{\square} \mathscr{A} \to M \otimes \mathscr{A}$. Hence $N \underset{E}{\square} p$ is a

left $C$-colinear retraction of $N \underset{E}{\Box} \iota$, and there is a commutative diagram with (split) exact rows

$$\begin{tikzcd} 0 \arrow[r] & N \underset{E}{\Box} M \underset{C}{\Box} \mathscr{A} \arrow[r, shift left, "N \underset{E}{\Box} \iota"] & N \underset{E}{\Box} M \otimes \mathscr{A} \arrow[l, shift left, "N \underset{E}{\Box} p"] \\ 0 \arrow[r] & \mathscr{A} \arrow[u, "\simeq"] \arrow[r, "{}^{\mathscr{A}}\rho"] & C \otimes \mathscr{A} \arrow[u, "\sigma_E \otimes \mathscr{A}"'] , \end{tikzcd}$$

from which a left $C$-colinear retraction of ${}^{\mathscr{A}}\rho$ is constructed. □

The main result of this section is contained in the following

**Theorem 3.11.** *Let $\mathscr{A}$ be a $C$-ring and $M$ a right $\mathscr{A}$-module which is a quasi-finite injector as a right $C$-comodule. Set $N = h_C(M,C)$ and $S = E_{\mathscr{A}}(M)$. View $N \otimes M$ and $N \underset{S}{\Box} M$ as left $\mathscr{A}$-modules with the left action as in Lemma 3.1. Let $\beta$ be as in Proposition 3.5.*

1. *The following statements are equivalent*
   (a) *there exists a left $\mathscr{A}$-module map $\chi : N \otimes M \to \mathscr{A}$ such that $\chi \circ \beta = \mathscr{A}$ (i.e. $\beta : \mathscr{A} \to N \otimes M$ is a split monomorphism of left $\mathscr{A}$-modules);*
   (b) *$M$ is a principal Galois $\mathscr{A}$-module.*
2. *The following statements are equivalent*
   (a) *there exists a left $\mathscr{A}$-module map $\hat{\chi} : N \underset{S}{\Box} M \to \mathscr{A}$ such that $\hat{\chi} \circ \beta = \mathscr{A}$ (i.e. $\beta : \mathscr{A} \to N \underset{S}{\Box} M$ is a split monomorphism of left $\mathscr{A}$-modules);*
   (b) *$M$ is a Galois $\mathscr{A}$-module.*

*Proof.*
(1) (a) $\Rightarrow$ (b) Suppose that there exists a left $\mathscr{A}$-module retraction $\chi$ of $\beta$. This means explicitly that, for all $a \in A$, $\chi(\sigma(a_{[-1]}) \triangleleft a_{[0]}) = a$, where $\sigma$ is the unit of the coendomorphism ring context. In particular, for $a = \eta_{\mathscr{A}}(c)$, this implies that, writing $\sigma(c) = c^{[1]} \otimes c^{[2]}$,

$$\eta_{\mathscr{A}}(c) = \chi(\sigma(c_{(1)}) \triangleleft \eta_{\mathscr{A}}(c_{(2)})) = \chi(c^{[1]} \otimes c^{[2]}{}_{[0]} \triangleleft \eta_{\mathscr{A}}(c^{[2]}{}_{[1]})) = \chi \circ \sigma(c),$$

where the first equality follows from the left $C$-colinearity of $\eta_{\mathscr{A}}$, the second by the right $C$-colinearity of $\sigma$ and the third by the unitality of the $\mathscr{A}$-action. Therefore,

$$\chi \circ \sigma = \eta_{\mathscr{A}}. \tag{3.1}$$

First we prove that $M$ is an injective left $S$-module, by constructing a left $S$-comodule retraction of the left $S$-coaction on $M$. Define a map $\delta : S \otimes M \to M$ by the commutative diagram

$$\begin{tikzcd} M \underset{C}{\Box} N \otimes M \arrow[r, "\pi_{\mathscr{A}} \otimes M"] \arrow[d, "M \underset{C}{\Box} \chi"'] & S \otimes M \arrow[d, "\delta"] \\ M \underset{C}{\Box} \mathscr{A} \arrow[r, "\overline{\rho_M}"] & M \end{tikzcd}$$

The map $\delta$ is well defined because $\chi$ is assumed to be a left $\mathscr{A}$-module map. By equation (3.1) and the unitality of the right $\mathscr{A}$-action we obtain, for all $m \in M$,

$$\delta \circ {}^{M}\rho(m) = m_{[0]} \triangleleft \chi(\sigma(m_{[1]})) = m_{[0]} \triangleleft \eta_{\mathscr{A}}(m_{[1]}) = m.$$

Hence $\delta$ is a retraction of the left coaction. Note that ${}^M\rho$ is right $\mathscr{A}$-linear since, for all $m\otimes a\in M\underset{C}{\Box}\mathscr{A}$ (summation suppressed),

$$\begin{aligned} {}^M\rho(m\lhd a) &= \pi_{\mathscr{A}}(m\lhd a_{[0]}\otimes {a_{[1]}}^{[1]})\otimes {a_{[1]}}^{[2]} = \pi_{\mathscr{A}}(m\otimes a_{[0]}\rhd {a_{[1]}}^{[1]})\otimes {a_{[1]}}^{[2]} \\ &= \pi_{\mathscr{A}}(m\otimes {a_{[-1]}}^{[1]})\otimes {a_{[-1]}}^{[2]}\lhd a_{[0]} = {}^M\rho(m)\lhd a, \end{aligned}$$

where the first equality holds because the $\mathscr{A}$-action is a right $C$-colinear. The second equality follows by the definition of $\pi_{\mathscr{A}}$ and the third by Lemma 3.2(2). To derive the last equality the fact that $m\otimes a\in M\underset{C}{\Box}\mathscr{A}$ was used. Now it is easy to see that, for all $m\otimes n\in M\underset{C}{\Box}N$ and $m'\in M$,

$$\begin{aligned} (S\otimes\delta)\circ(\Delta_S\otimes M)(\pi_{\mathscr{A}}(m\otimes n)\otimes m') &= \pi_{\mathscr{A}}(m_{[0]}\otimes {m_{[1]}}^{[1]})\otimes\delta(\pi_{\mathscr{A}}({m_{[1]}}^{[2]}\otimes n)\otimes m') \\ &= \pi_{\mathscr{A}}(m_{[0]}\otimes {m_{[1]}}^{[1]})\otimes {m_{[1]}}^{[2]}\lhd\chi(n\otimes m') \\ &= {}^M\rho(m)\lhd\chi(n\otimes m') = {}^M\rho(m\lhd\chi(n\otimes m')) \\ &= {}^M\rho\circ\delta(\pi_{\mathscr{A}}(m\otimes n)\otimes m'). \end{aligned}$$

To understand the first equality recall that the coproduct in $S=E_{\mathscr{A}}(M)$ is defined as $\Delta_S(\pi_{\mathscr{A}}(m\otimes n)) = \pi_{\mathscr{A}}(m_{[0]}\otimes {m_{[1]}}^{[1]})\otimes\pi_{\mathscr{A}}({m_{[1]}}^{[2]}\otimes n)$. The above calculation means that $\delta$ is a left $S$-comodule map and hence completes the proof that $M$ is an injective $S$-comodule. Now define $\hat\beta=\chi|_{N\underset{S}{\Box}M}$. As $\mathrm{Im}(\beta)\subset N\underset{S}{\Box}M$ and $\chi$ is a retraction of $\beta$, it is clear that $\hat\beta\circ\beta=\mathscr{A}$. To see that $\hat\beta$ is also a right inverse of $\beta$ take an element $n\otimes m\in N\underset{S}{\Box}M$ and compute

$$\begin{aligned} \beta\circ\hat\beta(n\otimes m) &= \sigma(\hat\beta(n\otimes m)_{[-1]})\lhd\hat\beta(n\otimes m)_{[0]} = \sigma(n_{[-1]})\lhd\hat\beta(n_{[0]}\otimes m) \\ &= n\otimes m_{[0]}\lhd\hat\beta(\sigma(m_{[1]})) = n\otimes m_{[0]}\lhd\eta_{\mathscr{A}}(m_{[1]}) = n\otimes m. \end{aligned}$$

The second equality is because $\chi$ is left $\mathscr{A}$-linear, which demands that it is left $C$-colinear. To justify the third equality, remember that $n\otimes m\in N\underset{S}{\Box}M$ and so, with coactions as in Corollary 3.4, $(N\otimes\pi_{\mathscr{A}}\otimes M)[\sigma(n_{[-1]})\otimes n_{[0]}\otimes m - n\otimes m_{[0]}\otimes\sigma(m_{[1]})]=0$. Since $\chi$ (and hence also $\hat\beta$) is left $\mathscr{A}$-linear, we can apply $(N\otimes\overline{\rho_M})\circ(N\otimes M\otimes\hat\beta)$ to this equality, thus obtaining the third equality in the above calculation. The fourth equality follows by equation (3.1) and the final equality by the unitality of the right $\mathscr{A}$-action. Thus $\hat\beta$ is the required inverse of $\beta$ and we conclude that $M$ is a principal Galois $\mathscr{A}$-module.

(1) (b)$\Rightarrow$ (a) Assume that $M$ is a principal Galois $\mathscr{A}$-module and let $\delta: S\otimes M\to M$ be an $S$-comodule retraction of ${}^M\rho$, i.e., $\delta\circ{}^M\rho=M$. We can construct a left $\mathscr{A}$-linear retraction for $\beta$ by making the following composition

$$\chi: N\otimes M\xrightarrow{\rho^N\otimes M} N\otimes S\otimes M\xrightarrow{N\otimes\delta} N\underset{S}{\Box}M\xrightarrow{\beta^{-1}}\mathscr{A}$$

Note that the image of the first two compositions is in $N\underset{S}{\Box}M$ because $\delta$ is left $S$-colinear. Note further that $\chi$ is left $\mathscr{A}$-linear, since ${}^N\rho$ is left $\mathscr{A}$-linear (by an argument similar to the proof of right $\mathscr{A}$-linearity of $\rho^M$ in (1) (a) $\Rightarrow$ (b)). Furthermore

$$\chi\circ\beta=\beta^{-1}\circ(N\otimes\delta)\circ(\rho^N\otimes M)\circ\beta=\beta^{-1}\circ(N\otimes\delta)\circ(N\otimes{}^M\rho)\circ\beta=\beta^{-1}\circ\beta=\mathscr{A},$$

where the second equality follows by the fact that $\mathrm{Im}(\beta)\in N\underset{S}{\Box}M$. Thus $\chi$ is the required retraction of $\beta$.

(2) That (b) implies (a) is obvious. For the converse use the same method as in the proof of the bijectivity of $\beta$ (1) (a) $\Rightarrow$ (b). $\square$

Theorem 3.11, which can be understood as a dual version of [7, Theorem 4.4], is the main result of the present paper. We will we use it in Section 5 to derive a weak algebra-Galois version of Schneider's Theorem II.

3.3. **A Galois connection.** The aim of this subsection is to construct a Galois connection associated to a matrix $C$-ring, following the method recently employed in the case of corings in [19]. Throughout this subsection $M$ is a right $C$-comodule which is a quasi-finite injector, $N := h_C(M,C)$ and $E$ is the coendomorphism coalgebra $E = h_C(M,M) \simeq M \underset{C}{\square} N$. Furthermore, $\pi : E \to D$ is a coalgebra epimorphism and $\mathscr{A} = N \underset{D}{\square} M$ is the associated matrix $C$-ring (cf. proof of Theorem 2.5).

For any subcoideal $X \subseteq \ker \pi$ (or, equivalently, a subcoextension $E \twoheadrightarrow E/X \twoheadrightarrow D$) define a matrix $C$-ring

$$\mathscr{A}(X) := N \underset{E/X}{\square} M.$$

For any subcoideal $Y \subseteq X$, the coalgebra map $E/Y \to E/X$ induces an inclusion of $C$-rings $\mathscr{A}(Y) \subseteq \mathscr{A}(X)$. Note that $\mathscr{A}(0) = N \underset{E}{\square} M$ and $\mathscr{A}(\ker \pi) = \mathscr{A}$. In particular $Y \subseteq \ker \pi$ induces an inclusion of $C$-rings $\mathscr{A}(Y) \subseteq \mathscr{A}$.

**Lemma 3.12.** *For any subcoideal $X \subseteq \ker \pi$,*

$$\ker \pi_{\mathscr{A}(X)} \subseteq X,$$

*where $\pi_{\mathscr{A}(X)} : E \to E_{\mathscr{A}(X)}(M)$ is the surjection defining the $\mathscr{A}(X)$-coendomorphism coalgebra of $M$ (cf. Theorem 3.3).*

*Proof.* Write $\pi_X : E \to E/X$ for the canonical coalgebra epimorphism. In view of the form of actions of $N \underset{E/X}{\square} M$ on $M$ and $N$ in Proposition 2.2 and the definition of $E_{\mathscr{A}(X)}(M)$ in Theorem 3.3, $x$ is an element of $\ker \pi_{\mathscr{A}(X)}$ if and only if there exists $m \otimes n \otimes m' \otimes n' \in M \underset{C}{\square} N \underset{E/X}{\square} M \underset{C}{\square} N$ (summation suppressed for clarity), such that

$$x = \widehat{\tau}_E(m \otimes n) m' \otimes n' - m \otimes n \widehat{\tau}_E(m' \otimes n').$$

Note that the $E/X$-coactions on $M$ and $N$ are

$${}^M\!\rho(m) = \pi_X(m_{[0]} \otimes {m_{[1]}}^{[1]}) \otimes {m_{[1]}}^{[2]}, \qquad \rho^N(n) = {n_{[-1]}}^{[1]} \otimes \pi_X({n_{[-1]}}^{[2]} \otimes n_{[0]}),$$

where we write $\sigma_E(c) = c^{[1]} \otimes c^{[2]}$, for all $c \in C$. If $m \otimes n \otimes m' \otimes n' \in M \underset{C}{\square} N \underset{E/X}{\square} M \underset{C}{\square} N$, then

$$\begin{aligned}
\widehat{\tau}_E(m \otimes n) \pi_X(m' \otimes n') &= \widehat{\tau}_E(m \otimes n) \pi_X({m'}_{[0]} \otimes {{m'}_{[1]}}^{[1]}) \widehat{\tau}_E({{m'}_{[1]}}^{[2]} \otimes n') \\
&= \widehat{\tau}_E(m \otimes {n_{[-1]}}^{[1]}) \pi_X({n_{[-1]}}^{[2]} \otimes n_{[0]}) \widehat{\tau}_E(m' \otimes n') \\
&= \pi_X(m \otimes n) \widehat{\tau}_E(m' \otimes n').
\end{aligned}$$

The first and third equalities follow by the fact that $\widehat{\tau}_E$ is the counit of $E$, while the second equality if a consequence of the fact that the middle cotensor product is over $E/X$. Hence, if $x \in \ker \pi_{\mathscr{A}(X)}$, then $x \in \ker \pi_X = X$, as required. $\square$

In view of Lemma 3.12, for any $C$-subring $\mathscr{B} \subseteq \mathscr{A}$ we can define the subcoideal of $\ker \pi$,

$$\mathscr{X}(\mathscr{B}) := \ker \pi_{\mathscr{B}},$$

where $\pi_{\mathscr{B}} : E \to E_{\mathscr{B}}(M)$ is the surjection defining the $\mathscr{B}$-coendomorphism coalgebra of $M$ (cf. Theorem 3.3). Note that if $\mathscr{B} \subseteq \mathscr{B}'$ are $C$-subrings of $\mathscr{A}$, then $\mathscr{X}(\mathscr{B}) \subseteq \mathscr{X}(\mathscr{B}')$. Thus we have defined an order-reversing correspondence between partially ordered sets

$$\{C\text{-subrings of } \mathscr{A}\} \rightleftarrows \{\text{subcoideals of } \ker\pi\} \ ,$$

where the subcoideals are ordered by the relation $X' \leq X$ iff $X \subseteq X'$ and the $C$-subrings by inclusion. We now prove that this correspondence is a Galois connection.

**Proposition 3.13.** *For all $C$-subrings $\mathscr{B} \subseteq \mathscr{A}$ and subcoideals $X \subseteq \ker\pi$,*

*(1) $\mathscr{B} \subseteq \mathscr{A}(\mathscr{X}(\mathscr{B}))$, and $\mathscr{B} = \mathscr{A}(\mathscr{X}(\mathscr{B}))$ if and only if $M$ is a Galois $\mathscr{B}$-module;*
*(2) $\mathscr{X}(\mathscr{A}(X)) \subseteq X$, and $\mathscr{X}(\mathscr{A}(X)) = X$ if and only if $E_{\mathscr{A}(X)}(M) = E/X$.*

*Proof.* (1) Compute,

$$\mathscr{A}(\mathscr{X}(\mathscr{B})) = \mathscr{A}(\ker\pi_{\mathscr{B}}) = N \underset{E/\ker\pi_{\mathscr{B}}}{\Box} M = N \underset{E_{\mathscr{B}}(M)}{\Box} M.$$

By Lemma 3.12, there is a coalgebra map $E_{\mathscr{B}}(M) \to D$, and we can consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & N \underset{E_{\mathscr{B}}(M)}{\Box} M \arrow[r] & N \underset{D}{\Box} M \\
0 \arrow[r] & \mathscr{B} \arrow[r] \arrow[u, "\beta"] & \mathscr{A} \arrow[u, equal]
\end{tikzcd} ,$$

where $\beta$ is the map in Proposition 3.5 (with $\mathscr{B}$ in place of $\mathscr{A}$). An easy calculation reveals that, for all $b \in \mathscr{B}$, $\beta(b) = b$. Therefore, the diagram is commutative and $\beta$ is the required inclusion. By the definition of a Galois $\mathscr{B}$-module, the map $\beta$ is identity iff $M$ is Galois.

(2) Note that $\mathscr{X}(\mathscr{A}(X)) = \ker\pi_{\mathscr{A}(X)}$ and the assertion follows by Lemma 3.12 □

*Remark* 3.14. By setting $\mathscr{A} = \mathscr{B}$ in the diagram in the proof of the first part of Proposition 3.13, it is immediately apparent that $M$ is a Galois $\mathscr{A}$-module. Moreover this is true for any matrix $C$-ring arising naturally from a coalgebra epimorphism with domain $E$ (cf. proof of Theorem 2.5).

**Corollary 3.15.** *The Galois connection constructed in Proposition 3.13 establishes a one-to-one correspondence between $C$-subrings $\mathscr{B} \subseteq \mathscr{A}$ such that $M$ is Galois $\mathscr{B}$-module and subcoideals $X \subseteq \ker\pi$ such that $E_{\mathscr{A}(X)}(M) = E/X$.*

*Proof.* For any subcoideal $X \subseteq \ker\pi$, $M$ is a Galois $\mathscr{A}(X)$-module by Remark 3.14. On the other hand if $M$ is a Galois $\mathscr{B}$-module, then $\mathscr{X}(\mathscr{A}(\mathscr{X}(\mathscr{B}))) = \mathscr{X}(\mathscr{B})$ by the first part of Proposition 3.13. Therefore $\mathscr{X}(\mathscr{B})$ is a subcoideal of $\ker\pi$ satisfying the required property by the second part of Proposition 3.13. □

The Galois connection constructed in Proposition 3.13 establishes a correspondence between 'intermediate coextensions' $E \twoheadrightarrow B \twoheadrightarrow D$ and sub $C$-rings $\mathscr{B} \subseteq \mathscr{A}$ and can be understood as a dual version of the Galois connection for comatrix corings described in [19, Proposition 2.1]. The latter is a generalisation of a Galois connection for Sweedler corings introduced in [25, Proposition 6.1] as a straightforward extension of the correspondence in Sweedler's Fundamental Theorem [28, Theorem 2.1].

## 4. $C$-RINGS ASSOCIATED TO INVERTIBLE WEAK ENTWINING STRUCTURES.

Recall from [16] that a *(right-right) weak entwining structure* is a triple $(A, C, \psi_R)$, where $A$ is an algebra, $C$ a coalgebra, and $\psi_R : C\otimes A \to A\otimes C$ a k-linear map which, writing, $\psi_R(c\otimes a) = \sum_\alpha a_\alpha \otimes c^\alpha$, $(A\otimes\psi_R)\circ(\psi_R\otimes A)(c\otimes a\otimes b) = \sum_{\alpha,\beta} a_\alpha\otimes b_\beta\otimes c^{\alpha\beta}$, etc., satisfies the relations

$$\sum_\alpha (ab)_\alpha \otimes c^\alpha = \sum_{\alpha,\beta} a_\alpha b_\beta \otimes c^{\alpha\beta}, \tag{4.1}$$

$$\sum_\alpha a_\alpha \varepsilon_C(c^\alpha) = \sum_\alpha \varepsilon_C(c^\alpha) 1_\alpha a, \tag{4.2}$$

$$\sum_\alpha a_\alpha \otimes \Delta_C(c^\alpha) = \sum_{\alpha,\beta} a_{\alpha\beta} \otimes {c_{(1)}}^\beta \otimes {c_{(2)}}^\alpha, \tag{4.3}$$

$$\sum_\alpha 1_\alpha \otimes c^\alpha = \sum_\alpha \varepsilon_C({c_{(1)}}^\alpha) 1_\alpha \otimes c_{(2)}. \tag{4.4}$$

This is a generalisation of the notion of a (right-right) entwining structure [11], motivated by the representation theory of weak Hopf algebras (cf. [5], [3]). Associated to a weak entwining structure $(A, C, \psi_R)$ is the category $\mathbf{M}(\psi_R)_A^C$ of *right weak entwined modules*, i.e. vector spaces $M$ together with a right $A$-action $\rho_M$ and a right $C$-coaction $\rho^M$ such that

$$\rho^M \circ \rho_M = (\rho_M \otimes C)\circ(M\otimes\psi_R)\circ(\rho^M\otimes A) \tag{4.5}$$

Also associated to a (right-right) entwining structure $(A, C, \psi_R)$ are projections

$$\overline{p_R} : C\otimes A \to C\otimes A, \qquad \overline{p_R} = (C\otimes A\otimes \varepsilon_C)\circ(C\otimes\psi_R)\circ(\Delta_C\otimes A), \tag{4.6}$$

$$p_R : A\otimes C \to A\otimes C, \qquad p_R = (\mu_A\otimes C)\circ(A\otimes\psi_R)\circ(A\otimes C\otimes 1_A). \tag{4.7}$$

That these are projections follows by equations (4.3) (in the case of $\overline{p_R}$) and (4.1) (in the case of $p_R$). Note further that

$$\psi_R\circ\overline{p_R} = p_R\circ\psi_R = \psi_R. \tag{4.8}$$

As explained in [6], the projection $p_R$ can be used to associate an $A$-coring to a weak entwining structure. On the other hand, $\overline{p_R}$ is needed for associating a $C$-ring to $(A, C, \psi_R)$ as follows:

**Theorem 4.1.** *Let $(A, C, \psi_R)$ be a (right-right) weak entwining structure and let*

$$\mathscr{A} = \mathrm{Im}\,\overline{p_R} = \{\sum_{\alpha,i} {c^i}_{(1)}\otimes a^i_\alpha \varepsilon_C({{c^i}_{(2)}}^\alpha) \mid \sum_i a^i\otimes c^i \in A\otimes C\}.$$

*Then:*

1. *$\mathscr{A}$ is a $(C,C)$-bicomodule with the left coaction ${}^{\mathscr{A}}\rho := \Delta_C\otimes A$ and the right coaction $\rho^{\mathscr{A}} := (C\otimes\psi_R)\circ(\Delta_C\otimes A)$.*
2. *The $(C,C)$-bicomodule $\mathscr{A}$ is a $C$-ring with product*
$$\mu_{\mathscr{A}} : \mathscr{A} \mathop{\Box}_C \mathscr{A} \to \mathscr{A}, \qquad \sum_i c_i\otimes a_i\otimes c'_i\otimes a'_i \mapsto \sum_i c_i\otimes\varepsilon_C(c'_i) a_i a'_i,$$
*and unit*
$$\eta_{\mathscr{A}} : C\to\mathscr{A}, \qquad c\mapsto \overline{p_R}(c\otimes 1).$$
3. $\mathbf{M}_{\mathscr{A}} \equiv \mathbf{M}(\psi_R)_A^C$.

*Proof.* (1) That ${}^{\mathscr{A}}\rho$ is a left coaction follows immediately from properties of the comultiplication. For $\rho^{\mathscr{A}}$, using (4.3) note that, for all $a\in A$ and $c\in C$,

$$\sum_{\alpha}\rho^{\mathscr{A}}(c_{(1)}\otimes a_{\alpha}\varepsilon_C(c_{(2)}{}^{\alpha}))=\sum_{\alpha}c_{(1)}\otimes a_{\alpha}\otimes c_{(2)}{}^{\alpha}. \tag{4.9}$$

We aim to show that $\rho^{\mathscr{A}}\circ\overline{p_R}=(\overline{p_R}\otimes C)\circ\rho^{\mathscr{A}}\circ\overline{p_R}$; because $\overline{p_R}$ is a projection, this will imply that $\rho^{\mathscr{A}}(\mathscr{A})\subset\mathscr{A}\otimes C$. Applying $\overline{p_R}\otimes C$ to (4.9) we obtain

$$\begin{aligned}\sum_{\alpha}(\overline{p_R}\otimes C)\circ\rho^{\mathscr{A}}\circ\overline{p_R}(c\otimes a) &= \sum_{\alpha,\beta}c_{(1)}\otimes a_{\alpha\beta}\varepsilon_C(c_{(2)}{}^{\beta})\otimes c_{(3)}{}^{\alpha}\\ &= \sum_{\alpha}c_{(1)}\otimes a_{\alpha}\otimes c_{(2)}{}^{\alpha}=\rho^{\mathscr{A}}\circ\overline{p_R}(c\otimes a),\end{aligned}$$

where the second equality is by (4.3) and the third by (4.9). To see that $\rho^{\mathscr{A}}$ is counital simply apply $C\otimes A\otimes\varepsilon_C$ to (4.9). To complete the proof that $\rho^{\mathscr{A}}$ is a coaction only remains to prove that it is coassociative. Take any $c\otimes a\in C\otimes A$ and compute

$$\begin{aligned}(\rho^{\mathscr{A}}\otimes C)\circ\rho^{\mathscr{A}}\circ\overline{p_R}(c\otimes a) &= \sum_{\alpha}\rho^{\mathscr{A}}(c_{(1)}\otimes a_{\alpha})\otimes c_{(2)}{}^{\alpha}=\sum_{\alpha,\beta}c_{(1)}\otimes a_{\alpha\beta}\otimes c_{(2)}{}^{\beta}\otimes c_{(3)}{}^{\alpha}\\ &=\sum_{\alpha}c_{(1)}\otimes a_{\alpha}\otimes c_{(2)}{}^{\alpha}{}_{(1)}\otimes c_{(2)}{}^{\alpha}{}_{(2)}=(\mathscr{A}\otimes\Delta_C)\circ\rho^{\mathscr{A}}\circ\overline{p_R}(c\otimes a),\end{aligned}$$

where the first and last equalities follow by (4.9) and the third by (4.3). Using the coassociativity of the coproduct one easily checks that left and right coactions commute with each other, thus making $\mathscr{A}$ into a $(C,C)$-bicomodule, as claimed.

(2) The map $\mu_{\mathscr{A}}$ is obviously left $C$-colinear. A simple calculation, which uses (4.9), confirms that $\mu_{\mathscr{A}}$ is also a right $C$-comodule map. Similarly, $\eta_{\mathscr{A}}$ is obviously left $C$-colinear. Using (4.4) and (4.9) we immediately find

$$\overline{p_R}(c_{(1)}\otimes 1)\otimes c_{(2)}=\sum_{\alpha}c_{(1)}\otimes 1_{\alpha}\otimes c_{(2)}{}^{\alpha}=\rho^{\mathscr{A}}\circ\overline{p_R}(1\otimes c),$$

hence $\eta_{\mathscr{A}}$ is right $C$-colinear as well. A straightforward calculation proves that $\mu_{\mathscr{A}}$ is associative and unital.

(3) Let $\Psi:\mathbf{M}_{\mathscr{A}}\to\mathbf{M}(\psi_R)_A^C$ be the map which leaves each $\mathscr{A}$-module unchanged as a $C$-comodule, but which changes the right $\mathscr{A}$-action $\overline{\rho_M}$ into a map $\Psi(\overline{\rho_M}):M\otimes A\to M$, $\Psi(\overline{\rho_M})=\overline{\rho_M}\circ(M\underset{C}{\Box}\overline{p_R})\circ(\rho^M\otimes A)$, which will presently be shown to be a right $A$-action for which $M$ is an entwined module. Unitality follows easily as

$$\Psi(\overline{\rho_M})(m\otimes 1)=\overline{\rho_M}\circ(M\underset{C}{\Box}\overline{p_R})(m_{[0]}\otimes m_{[1]}\otimes 1)=\overline{\rho_M}\circ(M\underset{C}{\Box}\eta_{\mathscr{A}})\circ\rho^M(m)=m,$$

where the second equality is by the definition of the unit and last equality comes from the unitality of an $\mathscr{A}$-action. For associativity, take any $a, a' \in A$ and $m \in M$ and compute

$$\begin{aligned}
&\Psi(\overline{\rho_M})\circ(\Psi(\overline{\rho_M})\otimes A)(m\otimes a\otimes a') \\
&\quad= \sum_\alpha \overline{\rho_M}\circ(M\underset{C}{\Box}\overline{p_R})(\rho^M\circ\overline{\rho_M}(m_{[0]}\otimes m_{[1]}\otimes a_\alpha\varepsilon_C({m_{[2]}}^\alpha))\otimes a') \\
&\quad= \sum_\alpha \overline{\rho_M}\circ(M\underset{C}{\Box}\overline{p_R})((\overline{\rho_M}\otimes C)(m_{[0]}\otimes m_{[1]}\otimes a_\alpha\otimes {m_{[2]}}^\alpha)\otimes a') \\
&\quad= \sum_\alpha \overline{\rho_M}\circ(\overline{\rho_M}\underset{C}{\Box}\mathscr{A})(m_{[0]}\otimes(m_{[1]}\otimes a_\alpha)\otimes\overline{p_R}({m_{[2]}}^\alpha\otimes a')) \\
&\quad= \sum_\alpha \overline{\rho_M}\circ(M\underset{C}{\Box}\mu_{\mathscr{A}})(m_{[0]}\otimes(m_{[1]}\otimes a_\alpha)\otimes\overline{p_R}({m_{[2]}}^\alpha\otimes a')) \\
&\quad= \sum_{\alpha,\beta} \overline{\rho_M}(m_{[0]}\otimes m_{[1]}\otimes a_\alpha a'_\beta\varepsilon_C({{m_{[2]}}^\alpha}_{(1)})\varepsilon_C({{{m_{[2]}}^\alpha}_{(2)}}^\beta)) \\
&\quad= \sum_\alpha \overline{\rho_M}(m_{[0]}\otimes m_{[1]}\otimes(aa')_\alpha\varepsilon_C({m_{[2]}}^\alpha)) = \Psi(\overline{\rho_M})(m\otimes aa').
\end{aligned}$$

Here the second equality is from the right $C$-colinearity of the map $\overline{\rho_M}$ and the equality (4.9). The fourth equality comes from the associativity of $\overline{\rho_M}$. The penultimate equality follows from the definition of a counit and (4.1). Next we check that this right action makes $M$ an entwined module:

$$\begin{aligned}
(\Psi(\overline{\rho_M})\otimes C)\circ(M\otimes\psi_R)\circ(\rho^M\otimes A) &= (\overline{\rho_M}\otimes C)\circ(M\otimes C\otimes\psi_R)\circ(M\otimes\Delta_C\otimes A)\circ(\rho^M\otimes A) \\
&= \overline{\rho_M}\circ(M\underset{C}{\Box}\rho^{\mathscr{A}}\circ\overline{p_R})\circ(\rho^M\otimes A) \\
&= \rho^M\circ\overline{\rho_M}\circ(M\underset{C}{\Box}\overline{p_R})\circ(\rho^M\otimes A) = \rho^M\circ\Psi(\overline{\rho_M}),
\end{aligned}$$

where the first equality follows by the coassociativity of a coaction, the definition of a counit and (4.3), the second by (4.9) and penultimate equality by the colinearity of $\overline{\rho_M}$.

Given a morphism $f : M \to N$ in $\mathbf{M}_{\mathscr{A}}$, we define $\Psi(f) = f$. Using the $C$-colinearity of $f$ and that $\overline{\rho_N}\circ(f\underset{C}{\Box}\mathscr{A}) = f\circ\overline{\rho_M}$, one easily finds that the map $f$ is also right $A$-linear, when $M$ and $N$ are viewed as $A$-modules with actions $\Psi(\overline{\rho_M})$ and $\Psi(\overline{\rho_N})$ respectively. Thus $\Psi$ is a functor.

In the other direction, define $\Theta : \mathbf{M}(\psi)_A^C \to \mathbf{M}_{\mathscr{A}}$ to be the map which leaves each entwined module $M$ unchanged as a $C$-comodule, but which changes the right $A$-action $\rho_M$ into a map $\Theta(\rho_M) : M\underset{C}{\Box}\mathscr{A} \to M$ defined as $\Theta(\rho_M) = \rho_M\circ(M\otimes\varepsilon_C\otimes A)$. Since $M\underset{C}{\Box}\mathscr{A} = (M\underset{C}{\Box}\overline{p_R})(M\underset{C}{\Box}C\otimes A)$, all elements of $M\underset{C}{\Box}\mathscr{A}$ are linear combinations of $x = \sum_\alpha m_{[0]}\otimes m_{[1]}\otimes a_\alpha\varepsilon_C({m_{[2]}}^\alpha)$ with $a \in A$ and $m \in M$. In view of the fact that $M$ is an entwined module, $\Theta(\rho_M)(x) = ma$. From this, the unitality and associativity of $\Theta(\rho_M)$ easily follow. The right $C$-colinearity of $\Theta(\rho_M)$ is confirmed by the following simple calculation that uses that $M$ is an entwined module and equation (4.9):

$$\begin{aligned}
\rho^M\circ\Theta(\rho_M)(x) &= \sum_\alpha m_{[0]}a_\alpha\otimes {m_{[1]}}^\alpha = (\rho_M\otimes C)\circ(M\otimes\varepsilon_C\otimes A\otimes C)\circ(M\otimes\rho^{\mathscr{A}})(x) \\
&= (\Theta(\rho_M)\otimes C)\circ(M\otimes\rho^{\mathscr{A}})(x).
\end{aligned}$$

Given a morphism $f : M \to N$ in $\mathbf{M}(\psi)_A^C$, define $\Theta(f) = f$. Then $\Theta(f)$ is obviously right $C$-colinear and is right $\mathscr{A}$-linear by the definition of the $\mathscr{A}$-action and the $A$-linearity of $f$.

Since the composition in both categories is provided by the composition in the category of vector spaces, $\Theta : \mathbf{M}(\psi)_A^C \to \mathbf{M}_{\mathscr{A}}$ is a functor.

Finally, note that for all $M \in \mathbf{M}(\psi)_A^C$, $m \in M$ and $a \in A$,

$$\Psi(\Theta(\rho_M))(m \otimes a) = \sum_\alpha \rho_M(m_{[0]} \otimes a_\alpha \varepsilon_C(m_{[1]}{}^\alpha)) = (ma)_{[0]} \varepsilon_C((ma)_{[1]}) = \rho_M(m \otimes a),$$

where the second equality follows by the fact that $M$ is an entwined module. On the other hand, taking $M \in \mathbf{M}_{\mathscr{A}}$ and applying $(\Theta(\Psi(\overline{\rho_M}))$ to $x = \sum_\alpha m_{[0]} \otimes m_{[1]} \otimes a_\alpha \varepsilon_C(m_{[2]}{}^\alpha)$ one immediately obtains that $(\Theta(\Psi(\overline{\rho_M}))(x) = \overline{\rho_M}(x)$. Therefore, $\Psi$ and $\Theta$ are inverse isomorphisms of the categories, as required. $\square$

As explained in [6, Example 2.4], there is a weak entwining structure associated to any weak coalgebra-Galois extension. Dually, there is a weak entwining structure associated to a *weak algebra-Galois coextension* as described in the following

**Example 4.2.** Let $A$ be an algebra, $C$ be a coalgebra and a right $A$-module with the action $\rho_C$. Define the coideal

$$I = \{(ca)_{(1)} \alpha((ca)_{(2)}) - c_{(1)} \alpha(c_{(2)} a) | a \in A, c \in C, \alpha \in \mathrm{Hom}(C,k)\},$$

let $B = C/I$ and let

$$\overline{\beta} : C \otimes A \to C \underset{B}{\Box} C, \qquad \overline{\beta} := (C \otimes \rho_C) \circ (\Delta_C \otimes A).$$

View $C \underset{B}{\Box} C$ as an object of ${}^C\mathbf{M}_A$ in the obvious way. Now suppose that $C \twoheadrightarrow B$ is a *weak algebra-Galois coextension*, i.e. that there exists a morphism $\overline{\chi} : C \underset{B}{\Box} C \to C \otimes A$ in ${}^C\mathbf{M}_A$ such that $\overline{\beta} \circ \overline{\chi} = C \underset{B}{\Box} C$. Let $\omega : C \underset{B}{\Box} C \to A$, $\omega := (\varepsilon_C \otimes A) \circ \overline{\chi}$ be the cotranslation map. Define

$$\psi_R : C \otimes A \to A \otimes C, \qquad \psi_R := (\omega \otimes C) \circ (C \otimes \Delta_C) \circ \overline{\beta}.$$

Then $(A, C, \psi_R)$ is a (right-right) weak entwining structure. Moreover $\psi_R$ is the unique weak entwining map such that $C \in \mathbf{M}(\psi_R)_A^C$ with structure maps $\Delta_C$ and $\rho_C$. This example can be proven along the same lines as [9, Theorem 3.5].

*Remark* 4.3. If $C \in \mathbf{M}(\psi_R)_A^C$, then the definition of $I$ coincides with that of $I_\kappa$ in the definition of a Galois $C$-ring (Definition 3.8), where $\kappa$ is the restriction of $\varepsilon_C \circ \rho_C$ to $\mathscr{A}$.

*Remark* 4.4. If $\mathscr{A}$ is a $C$-ring associated to a weak entwining structure, then $\mathscr{A}$ is a Galois $C$-ring iff $\overline{\beta}|_{\mathscr{A}} : \mathscr{A} \to C \underset{B}{\Box} C$ is a bijection.

A connection between weak algebra-Galois coextensions and Galois $C$-rings (hence also Galois $\mathscr{A}$-modules) is provided by the following

**Proposition 4.5.** *The $C$-ring associated to the weak entwining structure in Example 4.2 is a Galois $C$-ring. Conversely, if the $C$-ring associated to a weak entwining structure $(A, C, \psi_R)$ is a Galois $C$-ring, then $C$ is a weak algebra-Galois coextension.*

*Proof.* If $\mathscr{A}$ is the $C$-ring associated to the weak entwining structure in Example 4.2, then $\mathscr{A} = \mathrm{Im}(\overline{\chi} \circ \overline{\beta})$. Since $\overline{\beta} \circ \overline{\chi} = C \underset{B}{\Box} C$, the map $\overline{\beta}\,|_{\mathscr{A}}$ is a bijection. Therefore, by Remark 4.4, $\mathscr{A}$ is a Galois $C$-ring. Conversely if $\mathscr{A}$ is a Galois $C$-ring and associated to a weak entwining structure then by Remark 4.4, $\overline{\beta}|_{\mathscr{A}} : \mathscr{A} \to C \underset{B}{\Box} C$ is a bijection, furthermore it is clear from the definition of $\overline{\beta}$ that it is a morphism in ${}^C\mathbf{M}_A$. Now observe that the

composition of the maps $\overline{\beta}|_{\mathscr{A}}{}^{-1} : C\Box_B C \to \mathscr{A}$ and then the inclusion $\mathscr{A} \hookrightarrow C\otimes A$ is a morphism in ${}^C\mathbf{M}_A$ splitting $\overline{\beta}$. Therefore $C \twoheadrightarrow B$ is a weak algebra-Galois coextension. □

The notion of a (right-right) weak entwining structure has a left-handed counterpart. A *(left-left) weak entwining structure* is a triple $(A, C, \psi_L)$ consisting of an algebra $A$, a coalgebra $C$, and a $k$-linear map $\psi_L : A\otimes C \to C\otimes A$ which, writing, $\psi_L(a\otimes c) = \sum_E c_E \otimes a^E$, $\psi_L(a\otimes c) = \sum_F c_F \otimes a^F$ etc., satisfies the relations

$$\sum_E c_E \otimes (ab)^E = \sum_{E,F} c_{EF} \otimes a^F b^E, \tag{4.10}$$

$$\sum_E \varepsilon_C(c_E)a^E = \sum_E a\varepsilon_C(c_E)1^E, \tag{4.11}$$

$$\sum_E \Delta_C(c_E)\otimes a^E = \sum_{E,F} c_{(1)E}\otimes c_{(2)F}\otimes a^{EF}, \tag{4.12}$$

$$\sum_E c_E \otimes 1^E = \sum_E c_{(1)}\otimes \varepsilon_C(c_{(2)E})1^E. \tag{4.13}$$

Associated to a (left-left) entwining structure is the category of left entwined modules ${}^C_A\mathbf{M}(\psi_L)$ defined by the obvious modification of condition (4.5). Also, there are projections

$$\overline{p_L} : A\otimes C \to A\otimes C, \qquad \overline{p_L} = (\varepsilon_C\otimes A\otimes C)\circ(\psi_L\otimes C)\circ(A\otimes\Delta_C), \tag{4.14}$$

$$p_L : C\otimes A \to C\otimes A, \qquad p_L = (C\otimes\mu_A)\circ(\psi_L\otimes A)\circ(1\otimes C\otimes A). \tag{4.15}$$

Note that

$$\psi_L\circ\overline{p_L} = p_L\circ\psi_L = \psi_L. \tag{4.16}$$

In an analogous way as in Theorem 4.1, $\mathscr{B} = \mathrm{Im}\,\overline{p_L}$ is a $C$-ring, and ${}^C_A\mathbf{M}(\psi_L) \equiv {}_{\mathscr{B}}\mathbf{M}$. Note that the left and right $C$-coactions on $\mathscr{B}$ are given by ${}^{\mathscr{B}}\rho = (\psi_L\otimes C)\circ(A\otimes\Delta_C)$, $\rho^{\mathscr{B}} = A\otimes\Delta_C$, respectively. In the case of invertible weak entwining structures the $C$-rings associated to the left and right weak entwining structures are strictly related. Recall from [13]

**Definition 4.6.** An *invertible weak entwining structure* is a quadruple $(A, C, \psi_R, \psi_L)$ such that

(a) $(A, C, \psi_R)$ is a right-right weak entwining structure and $(A, C, \psi_L)$ is a left-left weak entwining structure;

(b) $\psi_R\circ\psi_L = p_R$ and $\psi_L\circ\psi_R = p_L$.

As observed in [1], if $(A, C, \psi_R, \psi_L)$ is an invertible weak entwining structure, then for all $c\in C$,

$$\sum_E \varepsilon_C(c_E)1^E = \sum_\alpha 1_\alpha\varepsilon_C(c^\alpha). \tag{4.17}$$

**Lemma 4.7** (cf. Proposition 1.5 in [1])**.** *Let $(A, C, \psi_R, \psi_L)$ be an invertible weak entwining structure. Then $\overline{p_R} = p_L$ and $\overline{p_L} = p_R$.*

*Proof.* To see that $\overline{p_R} = p_L$, take any $a\in A$ and $c\in C$, and compute

$$\begin{aligned}\overline{p_R}(c\otimes a) &= \sum_\alpha c_{(1)}\otimes a_\alpha\varepsilon_C(c_{(2)}{}^\alpha) = \sum_\alpha c_{(1)}\otimes\varepsilon_C(c_{(2)}{}^\alpha)1_\alpha a\\ &= \sum_E c_{(1)}\otimes\varepsilon_C(c_{(2)}{}_E)1^E a = \sum_E c_E\otimes 1^E a = p_L(c\otimes a),\end{aligned}$$

where the second equality follows by (4.2), the third by (4.17), and the fourth by (4.13). A similar calculation shows that $\overline{p_L} = p_R$. □

*Remark* 4.8. Lemma 4.7 shows that conditions (b) in the definition of an invertible weak entwining structure may be replaced with alternative conditions:

(b*) $\psi_R \circ \psi_L = \overline{p_L}$ and $\psi_L \circ \psi_R = \overline{p_R}$.

Note further that both $\mathscr{A}$ and $\mathscr{B}$ are not only $C$-rings but also $A$-corings.

**Proposition 4.9.** *Let $(A, C, \psi_R, \psi_L)$ be an invertible weak entwining structure and let $\mathscr{A} = \operatorname{Im} \overline{p_R}$ and $\mathscr{B} = \operatorname{Im} \overline{p_L}$ be the corresponding $C$-rings. Then the restrictions of the entwining maps*

$$\psi_L : \mathscr{B} \to \mathscr{A}, \qquad \psi_R : \mathscr{A} \to \mathscr{B}$$

*are inverse isomorphisms of $C$-rings.*

*Proof.* Since $\overline{p_R}$ and $\overline{p_L}$ are projections, the conditions (b*) in Remark 4.8 imply that the restrictions of $\psi_R$ and $\psi_L$ to $\operatorname{Im} \overline{p_R}$ and $\operatorname{Im} \overline{p_L}$ respectively, are inverse isomorphisms of vector spaces. Using (4.3) one easily finds that

$$(\psi_R \otimes C) \circ \rho^{\mathscr{A}} \circ \overline{p_R} = (A \otimes \Delta_C) \circ \psi_R \circ \overline{p_R} = \rho^{\mathscr{B}} \circ \psi_R \circ \overline{p_R},$$

where the second equality follows by the definition of the right $C$-coaction on $\mathscr{B}$. This shows that $\psi_R$ is right $C$-colinear. Similarly to show the $\psi_R$ is left $C$-colinear compute

$$\begin{aligned} {}^{\mathscr{B}}\rho \circ \psi_R \circ \overline{p_R} &= (\psi_L \otimes C) \circ (\psi_R \otimes C) \circ (C \otimes \psi_R) \circ (\Delta_C \otimes A) \circ \overline{p_R} \\ &= (\overline{p_R} \otimes C) \circ \rho^{\mathscr{A}} \circ \overline{p_R} = \rho^{\mathscr{A}} \circ \overline{p_R} = (C \otimes \psi_R) \circ {}^{\mathscr{A}}\rho \circ \overline{p_R}, \end{aligned}$$

where the first equality follows by the definition of ${}^{\mathscr{B}}\rho$ and property (4.3) and the second by the definition of an invertible weak entwining structure and the definition of the coaction ${}^{\mathscr{A}}\rho$. The third is a consequence of the fact that the image of $\rho^{\mathscr{A}}$ is in $\mathscr{A} \otimes C$ (compare the proof of Theorem 4.1(1)), and the last equality is immediate from the definitions of ${}^{\mathscr{A}}\rho$ and $\rho^{\mathscr{A}}$ in Theorem 4.1(1). Hence $\psi_R$ is a $(C, C)$-bicomodule map. Similarly one shows that $\psi_L$ is a $(C, C)$-bicomodule map. The unitality of $\psi_R$ is easily checked with the help of Lemma 4.7, (4.4) and (4.17),

$$\psi_R \circ \eta_{\mathscr{A}}(c) = \psi_R \circ \overline{p_R}(c \otimes 1) = \psi_R(c \otimes 1) = \sum_\alpha \varepsilon_C({c_{(1)}}^\alpha) 1_\alpha \otimes c_{(2)} = \overline{p_L}(1 \otimes c) = \eta_{\mathscr{B}}(c).$$

Since $\mathscr{A} \underset{C}{\square} \mathscr{A} = (\overline{p_R} \underset{C}{\square} \overline{p_R})(C \otimes A \underset{C}{\square} C \otimes A)$, it suffices to check the multiplicativity of $\psi_R$ on elements of the form

$$x = \sum_\alpha \overline{p_R}(c_{(1)} \otimes a_\alpha) \otimes \overline{p_R}({c_{(2)}}^\alpha \otimes a') = \sum_{\alpha,\beta} c_{(1)} \otimes a_\beta \otimes {{c_{(2)}}^\beta}_{(1)} \otimes a'_\alpha \varepsilon_C({{c_{(2)}}^\beta}_{(2)}{}^\alpha).$$

The definition of product in $\mathscr{A}$ and properties (4.1) and (4.8) yield

$$\psi_R \circ \mu_{\mathscr{A}}(x) = \sum_{\alpha,\beta} \psi_R(c_{(1)} \otimes a_\beta a'_\alpha \varepsilon_C({c_{(2)}}^{\beta\alpha})) = \psi_R \circ \overline{p_R}(c \otimes aa') = \psi_R(c \otimes aa').$$

On the other hand, in view of (4.8) and conditions (4.1) and (4.3)

$$\begin{aligned} \mu_{\mathscr{B}} \circ (\psi_R \underset{C}{\square} \psi_R)(x) &= \sum_\alpha \mu_{\mathscr{B}} \circ (\psi_R \underset{C}{\square} \psi_R)(c_{(1)} \otimes a_\alpha \otimes {c_{(2)}}^\alpha \otimes a') \\ &= \sum_{\alpha,\beta,\gamma} a_{\alpha\beta} a'_\gamma \varepsilon_C({c_{(1)}}^\beta) \otimes {c_{(2)}}^{\alpha\gamma} = \psi_R(c \otimes aa'). \end{aligned}$$

Thus $\psi_R$ is multiplicative, hence a $C$-ring isomorphism as required. □

**Corollary 4.10.** *Let $(A, C, \psi_R, \psi_L)$ be an invertible weak entwining structure. If $C \in \mathbf{M}(\psi_R)_A^C$, then $C \in {}_A^C\mathbf{M}(\psi_L)$ with the action, for all $a \in A$, $c \in C$,*

$$ac = \sum_E c_{E\,(1)} \varepsilon_C(c_{E\,(2)} a^E).$$

*Proof.* To see this make the following chain of deductions. First, if $C \in \mathbf{M}(\psi_R)_A^C$, then $C \in \mathbf{M}_{\mathscr{A}}$ by Theorem 4.1. The corresponding right $\mathscr{A}$-action is, for all $c \otimes a \in \mathscr{A}$ (summation suppressed for clarity) and $c' \in C$,

$$c' \triangleleft (c \otimes a) = \varepsilon_C(c) c' a.$$

Since there is an obvious matrix ring context $(C, C, {}^C C^C, {}^C C^C, \sigma, \tau)$ (cf. Example 2.3 or the proof of Proposition 3.9), by Lemma 3.1 $C$ is a left $\mathscr{A}$-module with left $\mathscr{A}$-action

$$(c \otimes a) \triangleright c' = c_{(1)} \varepsilon_C(c_{(2)} a) \varepsilon_C(c').$$

By Proposition 4.9, $\psi_L : \mathscr{B} \to \mathscr{A}$ is an isomorphism of $C$-rings and so $C \in {}_{\mathscr{B}}\mathbf{M}$ with left $\mathscr{B}$-action

$$(a \otimes c) \triangleright c' = \sum_E (c_E \otimes a^E) \triangleright c' = \sum_E c_{E\,(1)} \varepsilon_C(c_{E\,(2)} a^E) \varepsilon_C(c').$$

Finally we use the correspondence ${}_{\mathscr{B}}\mathbf{M} \equiv {}_A^C\mathbf{M}(\psi_L)$ to view $C$ in ${}_A^C\mathbf{M}(\psi_L)$ with the left $A$-action as stated. □

## 5. Coextensions of self-injective algebras

In this section we start with an invertible weak entwining structure such that $C$ is a right entwined module and then use Theorem 3.11 to deduce a criterion for this coalgebra to be a weak $A$-Galois coextension. Since we will work in this setting, $S = E_{\mathscr{A}}(C)$ (where $\mathscr{A}$ is the $C$-ring associated to the (right-right) weak entwining structure) will be the same as $B_\kappa$, by the isomorphism given in the proof of Proposition 3.9. Moreover, as stated in Remark 4.3, $B_\kappa = B$ so for simplicity we shall henceforth denote all these objects by $B$.

**Proposition 5.1.** *Let $(A, C, \psi_R, \psi_L)$ be an invertible weak entwining structure such that $C$ is a right entwined module, and let $\mathscr{A}$ be the $C$-ring corresponding to $(A, C, \psi_R)$. View $C$ as a left $A$-module as in Corollary 4.10. Then $C \twoheadrightarrow B$ is a weak $A$-Galois coextension and $C$ is injective as a left $B$-comodule if and only if there exists a $k$-linear map $\hat{g} : C \otimes C \to A$ such that, for all $c \in C$ and $a \in A$,*

$$\sum_\alpha a_\alpha \hat{g}(c^\alpha \otimes c') = \sum_\alpha \hat{g}(a_\alpha c^\alpha \otimes c'), \tag{5.1}$$

*and*

$$\hat{g}(c_{(1)} \otimes c_{(2)} a) = \sum_\alpha a_\alpha \varepsilon_C(c^\alpha). \tag{5.2}$$

Since it is assumed in Proposition 5.1 that $C$ is a weak entwined module with an $A$-action $\rho_C$, $C$ is a right $\mathscr{A}$-module. In view of Proposition 4.5 and Proposition 3.9, to prove Proposition 5.1 we need to find criteria for $C$ to be a principal Galois $\mathscr{A}$-module. Setting $\kappa = \varepsilon_C \circ \rho_C$ in the construction of the proof of Proposition 3.9, we obtain a left $\mathscr{A}$-module structure on $C \otimes C$,

$$\overline{{}_{C\otimes C}\rho} : \mathscr{A} \otimes C \simeq \mathscr{A} \mathop{\Box}_C C \otimes C \to C \otimes C, \qquad c \otimes a \otimes c' \mapsto c_{(1)} \varepsilon_C(c_{(2)} a) \otimes c'.$$

In view of Theorem 3.11 we need to study $\mathscr{A}$-module retractions of $\beta$ (or $\overline{\beta}$). First we classify all candidates for such retractions.

**Lemma 5.2.** *Given an invertible weak entwining structure $(A,C,\psi_R,\psi_L)$ with $C\in\mathbf{M}(\psi_R)_A^C$, there is a bijective correspondence between left $\mathscr{A}$-linear maps $g:C\otimes C\to\mathscr{A}$ and k-linear maps $\hat{g}:C\otimes C\to A$ satisfying condition* (5.1)*.*

*Proof.* Note that, in view of the form of the left $A$-action in Corollary 4.10, the condition (5.1) is equivalent to

$$\sum_\alpha a_\alpha\hat{g}(c^\alpha\otimes c')=\hat{g}(c_{(1)}\otimes c')\varepsilon_C(c_{(2)}a). \tag{5.3}$$

Given a $k$-linear map $\hat{g}$ satisfying condition (5.1) define $g:C\otimes C\to\mathscr{A}$ as $g:=\overline{p_R}\circ(C\otimes\hat{g})\circ(\Delta_C\otimes C)$, so on elements $g(d\otimes d')=\sum_\alpha d_{(1)}\otimes\hat{g}(d_{(3)}\otimes d')_\alpha\varepsilon_C(d_{(2)}{}^\alpha)$. Using (4.2), (4.4) and condition (5.3) we obtain, for all $d,d'\in C$,

$$\sum_\alpha\hat{g}(d_{(2)}\otimes d')_\alpha\varepsilon_C(d_{(1)}{}^\alpha)=\sum_\alpha\varepsilon_C(d_{(1)}{}^\alpha)1_\alpha\hat{g}(d_{(2)}\otimes d')=\sum_\alpha 1_\alpha\hat{g}(d^\alpha\otimes d')=\hat{g}(d\otimes d'),$$

hence

$$g(d\otimes d')=d_{(1)}\otimes\hat{g}(d_{(2)}\otimes d'). \tag{5.4}$$

Next note that $\mathscr{A}\,\square_C\,C\otimes C$ consists of $k$-linear combinations of $\sum_\alpha c_{(1)}\otimes a_\alpha\otimes c_{(2)}{}^\alpha\otimes d$, with $a\in A$ and $c,d\in C$, and compute

$$\begin{aligned}\sum_\alpha g((c_{(1)}\otimes a_\alpha)\triangleright(c_{(2)}{}^\alpha\otimes d)) &= \sum_\alpha d(c_{(1)}\otimes\varepsilon_C(c_{(2)}a_\alpha)\varepsilon_C(c_{(3)}{}^\alpha)d)\\ &= g(c_{(1)}\otimes d)\varepsilon_C(c_{(2)}a)=c_{(1)}\otimes\hat{g}(c_{(2)}\otimes d)\varepsilon_C(c_{(3)}a)\\ &= \sum_\alpha c_{(1)}\otimes a_\alpha\hat{g}(c_{(2)}{}^\alpha\otimes d)=\sum_\alpha(c_{(1)}\otimes a_\alpha)g(c_{(2)}{}^\alpha\otimes d),\end{aligned}$$

where the second equality follows by the fact that $C$ is a weak entwined module, the third by (5.4), the fourth by condition (5.3). The final equality is a consequence of (5.4) and the definition of product in $\mathscr{A}$. This shows that $g$ is a left $\mathscr{A}$-module map.

For the converse, given a left $\mathscr{A}$-linear map $g:C\otimes C\to\mathscr{A}$ define $\hat{g}:C\otimes C\to A$ to be $\hat{g}:=(\varepsilon_C\otimes A)\circ g$. Observe that $\sum_\alpha d_{(1)}\otimes a_\alpha\otimes d_{(2)}{}^\alpha\otimes d'$ lies in $\mathscr{A}\,\square_C\,C\otimes C$ for all $a\in A$ and $d\in C$. Apply the map $\varepsilon_C\otimes A:\mathscr{A}\to A$ to the $\mathscr{A}$-linearity condition of $g$

$$\sum_\alpha(d_{(1)}\otimes a_\alpha)g(d_{(2)}{}^\alpha\otimes d')=\sum_\alpha g((d_{(1)}\otimes a_\alpha)\triangleright(d_{(2)}{}^\alpha\otimes d')),$$

and observe that $\varepsilon_C\otimes A$ is multiplicative with respect to the $C$-ring product in $\mathscr{A}$, to conclude that $\hat{g}$ satisfies the required condition (5.1).

It remains to show that the given correspondence is one-to-one. Clearly, applying $\varepsilon_C\otimes A$ to $g$ given in terms of $\hat{g}$ via equation (5.4), one obtains back $\hat{g}$. On the other hand, since $g$ is left $C$-colinear, $g=(C\otimes\varepsilon_C\otimes A)\circ(C\otimes g)\circ(\Delta_C\otimes C)$, thus establishing the converse correspondence. □

Using this lemma we are now able to prove Proposition 5.1.

*Proof.* (Proposition 5.1) Suppose that there is a map $\hat{g}:C\otimes C\to A$ satisfying (5.1) and (5.2). By Lemma 5.2 there is a corresponding left $\mathscr{A}$-linear map $g:C\otimes C\to\mathscr{A}$, $c\otimes c'\mapsto c_{(1)}\otimes\hat{g}(c_{(2)}\otimes c')$. The condition (5.2) ensures that $g$ is a retraction of $\beta$, hence $C\twoheadrightarrow B$ is a weak $A$-Galois coextension and $C$ is injective as a left $B$-comodule by Theorem 3.11.

Conversely, if $C \twoheadrightarrow B$ is a weak $A$-Galois coextension and $C$ is injective as a left $B$-comodule, then, by Theorem 3.11 there is a left $\mathscr{A}$-module retraction $g$ of $\beta$. The map $\hat{g} = (\varepsilon_C \otimes A) \circ g$ satisfies (5.1) (by Lemma 5.2) and (5.2) (since $g$ is a retraction of $\beta$). □

In the case where $\psi_R$ is a bijective entwining structure (non-weak!), $\psi_R$ is a bijective map (with the inverse $\psi_L$), hence the condition (5.1) means that $\hat{g}$ is left $A$-linear.

**Example 5.3.** Let $H$ be a Hopf algebra with bijective antipode $S$, and let $A$ be a right $H$-coideal subalgebra of $H$, i.e. $A$ is a subalgebra of $H$ and $\Delta_H(A) = A \otimes H$. In this case $(A, H, \psi_R)$, with

$$\psi_R : H \otimes A \to A \otimes H, \qquad h \otimes a \mapsto a_{(1)} \otimes ha_{(2)},$$

we have a bijective right entwining structure for which $H$ is an entwined module. The inverse of $\psi_R$ is

$$\psi_L : A \otimes H \to H \otimes A, \qquad a \otimes h \mapsto hS^{-1}(a_{(2)}) \otimes a_{(1)},$$

hence the induced left $A$-action on $H$ is $a \cdot h := hS^{-1}(a)$. Suppose that $A$ is a direct summand of $H$ as a left $A$-module (e.g. there is a strong connection in $H$, cf. [10, Theorem 2.5]), and let $p : H \to A$ be a left $A$-linear retraction of $A \subseteq H$. Define the map

$$\hat{g} : H \otimes H \to A, \qquad h \otimes h' \mapsto p(S(h)h').$$

Then the map $\hat{g}$ satisfies both (5.1) and (5.2), hence $H \twoheadrightarrow B$ is an $A$-Galois coextension and $H$ is injective as a left $B$-comodule. In this case $B = H/HA^+$, where $A^+ = A \cap \ker \varepsilon_H$.

As a concrete illustration of Example 5.3, take $H = \mathscr{O}(SU_q(2))$, the algebra of (polynomial) functions on the quantum group $SU_q(2)$ [32] and $A = \mathscr{O}(S^2_{q,s})$, the algebra of (polynomial) functions on the quantum two-sphere [24]. $\mathscr{O}(SU_q(2))$ is known to be a coalgebra-Galois extension of $\mathscr{O}(S^2_{q,s})$ with a strong connection (explicitly constructed in [12]). This implies that $\mathscr{O}(S^2_{q,s})$ is a direct summand of $\mathscr{O}(SU_q(2))$ as a left $\mathscr{O}(S^2_{q,s})$-module. The coinvariant coalgebra $B$ is spanned by countably many group-like elements (hence it can be identified with the Hopf algebra $\mathscr{O}(S^1) = k[Z, Z^{-1}]$). Consequently, $\mathscr{O}(SU_q(2))$ is an $\mathscr{O}(S^2_{q,s})$-Galois coextension of $B$ and it is injective as a $B$-comodule.

Proposition 5.1 can be used to characterise weak Galois coextensions of self-injective algebras.

**Theorem 5.4.** *Let $(A, C, \psi_R, \psi_L)$ be an invertible weak entwining structure such that $C$ is a right entwined module, and let $\mathscr{A}$ be the $C$-ring corresponding to $(A, C, \psi_R)$. Suppose that the map $\beta : \mathscr{A} \to C \otimes C$, $c \otimes a \mapsto c_{(1)} \otimes c_{(2)}a$ is injective. If $A$ is a left self-injective algebra, then $C \twoheadrightarrow B$ is a weak $A$-Galois coextension and $C$ is injective as a left $B$-comodule. Furthermore, if $A$ is a separable algebra, then $C$ is also $A$-equivariantly injective as a left $B$-comodule (i.e., $C$ is an injective left $B$ comodule and the corresponding coaction has a retraction in ${}^B\mathbf{M}_A$).*

*Proof.* Firstly view $\mathscr{A}$ as a left $A$-module by

$$A \otimes \mathscr{A} \xrightarrow{\psi_L \otimes A} C \otimes A \otimes A \xrightarrow{C \otimes \mu_A} \mathscr{A}.$$

This is easily seen to be well-defined, since $\mathscr{A} = \mathrm{Im}\overline{p_R} = \mathrm{Im} p_L$. Secondly, view $C \otimes C$ as a left $A$-module through the composition

$$A \otimes C \otimes C \xrightarrow{\psi_L \otimes C} \mathscr{A} \otimes C \xrightarrow{\overline{C \otimes C \rho}} C \otimes C,$$

i.e., use the left $A$-action in Corollary 4.10, $a\otimes c\otimes c' \mapsto ac\otimes c'$. Define the map

$$r : \mathscr{A} \to A, \qquad r(c\otimes a) = \varepsilon_C(c)a,$$

and observe that, for all $b \in A$ and $c\otimes a \in \mathscr{A}$ (summation suppressed for clarity),

$$r(b(c\otimes a)) = \sum_E \varepsilon_C(c_E)b^E a = \sum_E b\varepsilon_C(c_E)1^E a = b\varepsilon_C(c)a.$$

The second equality is by (4.11) and final equality since $c\otimes a \in \mathscr{A}$ implies that $\sum_E c_E \otimes 1^E a = p_L(c\otimes a) = c\otimes a$. Hence $r \in \mathrm{Hom}_{A-}(\mathscr{A}, A)$. Next we prove that the map $\beta : \mathscr{A} \to C\otimes C$ is also $A$-linear. This is done in a few steps. First, using (4.12) note that, for all $a \in A$ and $c \in C$,

$$\sum_E c_{(1)}{}_E\varepsilon_C(a^E c_{(2)}) = ac. \tag{5.5}$$

On the other hand, since $C \in \mathbf{M}(\psi)^C_A$ and $\psi_L \circ \psi_R = \overline{p_R}$, we find that

$$\begin{aligned}\sum_\alpha \varepsilon_C(a_\alpha c^\alpha{}_{(1)})c^\alpha{}_{(2)} &= \sum_{\alpha,E} \varepsilon_C(c^\alpha{}_{(1)}{}_E a_\alpha{}^E)c^\alpha{}_{(2)} = \sum_{\alpha,\beta,E} \varepsilon_C(c_{(1)}{}^\beta{}_E a_{\alpha\beta}{}^E)c_{(2)}{}^\alpha \\ &= \sum_{\alpha,\beta} \varepsilon_C(c_{(1)}a_{\alpha\beta})\varepsilon_C(c_{(2)}{}^\beta)c_{(3)}{}^\alpha = ca,\end{aligned}$$

where the second equality follows by the definition of the left $A$-action in Corollary 4.10. We can combine this way of expressing of right $A$-action on $C$ in terms of the left $A$-action with the equality $\psi_R \circ \psi_L = \overline{p_L}$ and the fact that $C \in {}^C_A\mathbf{M}(\psi_L)$, to find that, for all $c \in C$ and $a \in A$,

$$\sum_E c_E a^E = \sum_{\alpha,E} \varepsilon_C(a^E{}_\alpha c_E{}^\alpha{}_{(1)})c_E{}^\alpha{}_{(2)} = \sum_E \varepsilon_C(a^E c_{(2)})\varepsilon_C(c_{(1)}{}_E) = \varepsilon_C(ac_{(1)})c_{(2)}. \tag{5.6}$$

Therefore, for all $a, b \in A$ and $c \in C$,

$$\begin{aligned}\beta(bp_L(c\otimes a)) &= \sum_E c_{E\,(1)}\otimes c_{E\,(2)}b^E a = \sum_{E,F} c_{(1)}{}_E\otimes c_{(2)}{}_F b^{EF} a \\ &= \sum_E c_{(1)}{}_E\varepsilon_C(b^E c_{(2)})\otimes c_{(3)}a = bc_{(1)}\otimes c_{(2)}a = b\beta(p_L(c\otimes a)),\end{aligned}$$

where the second equality is by (4.12), the third by (5.6) and the fourth by (5.5). This proves that $\beta$ is a left $A$-linear map, and thus, in view of the self-injectivity of $A$, we are led to an exact sequence

$$\mathrm{Hom}_{A-}(C\otimes C, A) \xrightarrow{\beta^*} \mathrm{Hom}_{A-}(\mathscr{A}, A) \to 0$$

and so there exists $\hat{g} \in \mathrm{Hom}_{A-}(C\otimes C, A)$ s.t. $\beta^* \circ \hat{g} = \hat{g}\circ\beta = r$. By construction, $\hat{g}$ satisfies condition (5.2) and it is left $A$-linear, hence (5.1) holds. By Proposition 5.1, $C \twoheadrightarrow B$ is a weak $A$-Galois coextension and $C$ is injective as a left $B$-comodule.

Now suppose furthermore that $A$ is a separable algebra and let $e = e_1 \otimes e_2 \in A\otimes A$ denote the separability element (summation suppressed). To show that $C$ is $A$-equivariantly injective as a left $B$-module we need to show that there exists a retraction of the left $B$-coaction, given in Corollary 3.4, in ${}^B\mathbf{M}_A$. The injectivity of $C$ as a left $B$-module guarantees that there is a left $B$-colinear map $\hat{\lambda} : B\otimes C \to C$ such that $\hat{\lambda}\circ{}^C\rho = C$. From this we can construct

$$\lambda : B\otimes C \to C, \qquad \lambda = \rho_C \circ (\hat{\lambda}\otimes A)\circ(B\otimes\rho_C\otimes A)\circ(B\otimes C\otimes e).$$

Now observe that $\rho_C : C \otimes A \to C$ is a left $B$-comodule map because ${}^C\rho : C \to B \otimes C$, given in Corollary 3.4, is right $\mathscr{A}$-linear and the correspondence given in the third part of Theorem 4.1 allows the right $A$-action on $C$ to be viewed as some right $\mathscr{A}$-action. With this in mind it is clear that $\sigma$ is left $B$-colinear, since it is a composition of $B$-colinear maps. That it is a right $A$-linear map follows by the fact that $ea = ae$, for all $a \in A$. It only remains to show that this map is indeed a retraction for the left $B$-coaction. Just compute

$$\begin{aligned} \lambda \circ {}^C\rho(c) &= \hat{\lambda}(c_{[-1]} \otimes c_{[0]} e_1) e_2 \\ &= \hat{\lambda}((ce_1)_{[-1]} \otimes (ce_1)_{[0]}) e_2 \\ &= ce_1 e_2 = c, \end{aligned}$$

where the second equality follows from the left $B$-colinearity of the right $A$-action, the third because $\hat{\lambda}$ was chosen to be a splitting of the coaction and the final equality from the properties of the separability element. □

Theorem 5.4 is a dual version of [13, Theorems 5.1, 6.1], thus a dualisation of each in the long chain of generalisations of the Kreimer-Takeuchi theorem [23, Theorem 1.7] for Hopf-Galois extensions. In particular, in its self-injective part, the non-weak case corresponds to [26, Theorem 3.1], the proof of which lends the idea for the proof of Theorem 5.4. Since any quasi-Frobenius algebra is self-injective, Theorem 5.4 implies also a dual version of [2, Theorem 3.1]. In particular, this is applicable to extensions of finite dimensional weak Hopf algebras. Any such weak Hopf algebra $H$ has a bijective antipode by [5, Theorem 2.10] thus the weak entwining structure $(H, C, \psi_R)$ corresponding to a right $H$-module coalgebra $C$ is invertible. Furthermore, a finite dimensional weak Hopf algebra is quasi-Frobenius by [5, Theorem 3.11]. Hence Theorem 5.4 implies that the injectivity of the canonical map $\beta$ is sufficient for a coextension $C$ of a finite dimensional weak Hopf algebra $H$ to be a weak Hopf-Galois coextension.

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